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Calibrating the frequency of tuning forks by means of Lissajous figures

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We produce Lissajous figures by modulating a laser beam along two perpendicular directions by means of two tuning forks. When the ratio of the resonant frequencies of the tuning forks is a small rational number, Lissajous figures are clearly resolved on a screen. Because the ratio of frequencies is never exactly a rational number, a phase drift of the Lissajous figures is observed as a function of time. If the period of this phase drift is measured, variations in the resonant frequencies as small as 0.01 Hz can be detected. © 2011 American Association of Physics Teachers.
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I. INTRODUCTION

Tuning forks have been employed by musicians as frequency standards for many years mainly because they are very stable, portable, and easy to use. The physics of the tuning fork is described in many general physics textbooks, and a more detailed description of the modes of vibration (including the nonlinear motion of the prongs) can be found in more specialized literature.¹⁻³

Jules Antoine Lissajous was interested in waves and developed around 1855 a method for displaying waves optically. His method consisted of reflecting a light beam from two tuning forks, vibrating at right angles to each other, with mirrors attached to their prongs. In this paper, we describe an experimental setup suitable for an introductory physics course, which resembles the original method of Lissajous.^{4,5}

Although Lissajous figures are familiar to undergraduate physics students, they are usually observed as steady-state figures on an oscilloscope. Our approach produces these figures by modulating the trajectory of a laser beam by two independent mechanical oscillators (tuning forks), showing that the concept of Lissajous figures is very general. An interesting question is how similar are the frequencies of otherwise identically tuning forks. Our experiments provide a way to determine differences as small as 0.01 Hz.

In this paper, we describe the basic equations governing the formation of these figures and show that important quantitative information can be obtained by carefully measuring the drift of the figures as a function of time. In particular, we show that our setup allows the resonant frequency of a tuning fork to be calibrated with an accuracy of 0.01%.

II. EXPERIMENT

The idea of our experiment is to produce Lissajous figures by modulating a laser beam in two perpendicular directions using two tuning forks that vibrate independently. We used two identical aluminum tuning fork sets from PASCO (SE-7342). Each set includes eight forks representing a full octave of frequencies. The experimental setup is shown in Fig. 1. The two tuning forks are mounted on a table at 90° to each other. The degree of orthogonality is not very important for our purpose, so was done just by eye. The distance between them was 60 cm.

The mass and resonance frequencies of the tuning forks used are listed in Table I. To reflect the laser beam efficiently, a mirror has to be attached to the end of the tuning forks. The mirror must be light because otherwise it could modify the resonance frequency of the tuning forks. We chose a (10 mm × 10 mm × 0.5 mm) piece of a Si(111) wafer.⁶ Because a wafer of 75 mm in diameter has a mass of 3 g, the mass of each small mirror can be estimated to be $\approx 0.066 \pm 0.004$ g.

Once the laser beam has been reflected by the tuning forks, it is projected onto a screen located 2–3 m away from the table. A typical run consisted of beating simultaneously both tuning forks, while the image projected on the screen was recorded using a video camera able to record 15 frames/s.

An initial characterization of the tuning forks was done to determine their actual resonant frequencies. After hitting a tuning fork with a rubber hammer, the sound was recorded with a microphone and analyzed using a PC sound card and the Data Acquisition Toolbox by MATLAB. The programs provided by MATLAB were used to obtain the frequency spectrum of the tuning fork by means of a fast Fourier transform (FFT). The sampling rate used (96 kHz) and recording time (10 s) allowed the determination of the resonant frequency within an accuracy better than 0.1 Hz. We determined the true resonance values corresponding to the two sets of tuning forks used (see Table II). We see that these frequencies differ by ± 0.5 –1 Hz from their nominal values. In the last column, we show the frequency measured for set 2 when the small mirror was attached to the end of the arm of the tuning fork. We see that the mass of 0.066 g is enough to modify the resonance frequency by 0.5–1 Hz.

The typical length of the tuning fork prong is 120 mm, with an uncertainty of ± 0.15 mm. From Table I, the mass of one arm can be estimated to be ≈ 25 g, which means that the uncertainty in the length corresponds to ± 0.03 g in the mass of each arm. Because this value is comparable to the mass of the small mirror used in our experiments, we conclude that the differences in the frequencies we measured for sets 1 and 2 (shown in Table II) can be mainly attributed to the small differences in the lengths of the tuning fork prongs.

We tried to modify systematically the resonance frequency of the tuning forks by adding a small mass at different positions from the end of the tuning fork. For this purpose, we



Fig. 1. Experimental setup used to produce Lissajous figures using two tuning forks. After being reflected by the two mirrors, the laser beam is projected onto a screen located 2–3 m from the table.

used a rubber band with mass of 0.42 ± 0.02 g. The result of this experiment is shown in Fig. 2 for the tuning fork with a nominal frequency of 384 Hz. By this procedure, a frequency variation of ≈ 8 Hz can be induced, which is large compared to the change induced by the small mirrors attached to the tuning forks.

III. THEORY

We briefly discuss the conditions that lead to the formation of stable Lissajous figures. The equations governing the movement of the light spot projected on the screen are

$$x = X_0 \cos(\omega_1 t), \quad (1)$$

$$y = Y_0 \cos(\omega_2 t + \delta), \quad (2)$$

where ω_1 and ω_2 are the angular frequencies of the horizontal (x axis) and vertical (y axis) tuning forks, respectively, X_0 and Y_0 are the corresponding amplitudes of the oscillations along the x and y axes, respectively, and δ is the constant phase between the two oscillations. The Lissajous figures appear when the movement is periodic with period T , leading to the formation of closed trajectories for the light spot projected on the screen. The conditions for the appearance of closed trajectories are

$$X_0 \cos(\omega_1 t) = X_0 \cos[\omega_1(t + T)], \quad (3)$$

$$Y_0 \cos(\omega_2 t + \delta) = Y_0 \cos[\omega_2(t + T) + \delta], \quad (4)$$

which implies that

Table I. Mass of the different tuning forks used in the present experiments. The frequencies quoted are the nominal ones provided by PASCO.

Frequency (Hz)	Mass (g)
256	68 ± 1
288	65 ± 1
320	63 ± 1
341.3	62 ± 1
384	59 ± 1
426.7	56 ± 1
480	54 ± 1
512	53 ± 1

Table II. Measured frequencies of the two sets of tuning forks used in the present experiments as compared to the nominal ones. The corresponding error bars are ± 0.04 Hz.

Nominal frequency (Hz)	Set 1 (Hz)	Set 2 (Hz)	Set 2-mirror (Hz)
256	255.6	255.6	256
288	288.6	287.8	287
341.3	341.3	340.5	340.6
384	383	382.6	381.6
426.7	427.7	427	426.5
480	481.2	479.7	478.5

$$\omega_1 T = 2\pi K_1, \quad (5)$$

$$\omega_2 T = 2\pi K_2, \quad (6)$$

where K_1 and K_2 are integers. The ratio of Eq. (5) to Eq. (6) gives

$$K \equiv \frac{K_1}{K_2} = \frac{\omega_1}{\omega_2}. \quad (7)$$

We see that Lissajous figures are observed only when the ratio of the two frequencies is a rational number K .⁴

It is interesting to consider the case for which K is close to but not exactly a rational number, or in other words, the case in which $\omega_2 = \tilde{\omega}_2 + \Delta\omega_2$, with $\Delta\omega_2 \ll \tilde{\omega}_2$. This case is described by

$$K = \frac{\omega_1}{\omega_2} = \frac{\omega_1}{\tilde{\omega}_2 + \Delta\omega_2}, \quad (8)$$

and the corresponding equations of motion are

$$x = X_0 \cos(\omega_1 t), \quad (9)$$

$$y = Y_0 \cos(\tilde{\omega}_2 t + \delta + \Delta\omega_2 t). \quad (10)$$

We define the effective phase factor $\delta_{\text{eff}} = \delta + \Delta\omega_2 t$. Note that δ_{eff} can be regarded as a slowly varying phase angle, which leads to a drift of the Lissajous figures as a function of time. The period of this drift contains information about $\Delta\omega_2$.

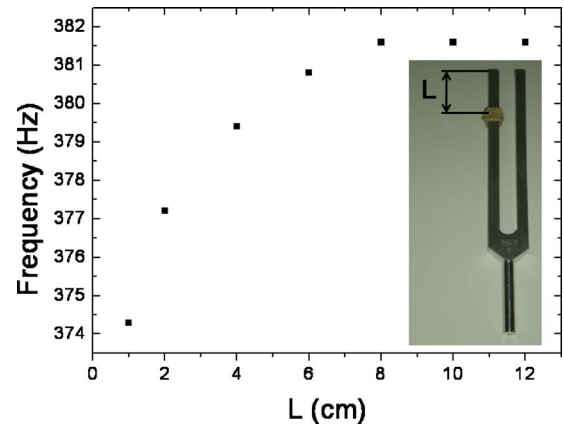


Fig. 2. Measured resonance frequency as a function of the position L of the rubber band, as defined in the inset, for the tuning fork of nominal frequency of 384 Hz.

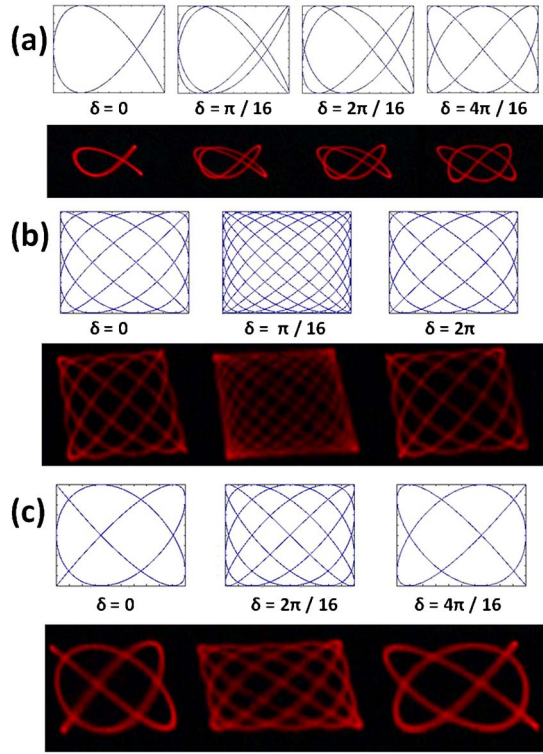


Fig. 3. A series of snapshots taken from a video of Lissajous figures (time increasing from left to right) corresponding to (a) $K=3/2$, (b) $9/8$, and (c) $5/4$. The corresponding figures predicted by the theory at different δ values are also shown.

IV. RESULTS

Figure 3 shows a series of snapshots taken from a video of Lissajous figures, which illustrate the results obtained for typical runs. The K values were obtained by fixing the horizontal tuning fork at 256 Hz and choosing the vertical fork at 384 Hz ($K=3/2$), 320 Hz ($K=5/4$), and 288 Hz ($K=9/8$). The evolution of the figures observed in the video is a consequence of the fact that δ_{eff} varies slowly with time, leading to a drift of the observed Lissajous figures.

As we can see from its definition, δ_{eff} contains information on the amount by which the ratio ω_1/ω_2 differs from a rational number. We can determine $\Delta\omega_2$ quantitatively by measuring the periodicity of δ_{eff} . The time T^* it takes for δ_{eff} to by 2π can be determined by monitoring the period T^* associated with the drift of the Lissajous figures. This time is given by $\delta_{\text{eff}}(t+T^*) = \delta_{\text{eff}}(t) + 2\pi$, which implies that

$$\delta + \Delta\omega_2(t + T^*) = \delta + \Delta\omega_2 t + 2\pi, \quad (11)$$

and therefore

$$\Delta\omega_2 = \frac{2\pi}{T^*}. \quad (12)$$

The meaning of T^* can be seen in Fig. 3(b) for $K=9/8$. We see that the same curve is observed after a time $t=T^*$ has elapsed, which corresponds to a variation of δ_{eff} by 2π . Equation (12) reveals that information on $\Delta\omega_2$ can be obtained by measuring T^* . Because typical values of the time T^* are in the range of 1–10 s, it can be measured with high accuracy even by recording videos with moderate resolution. We used rates as low as 15 frames/s.

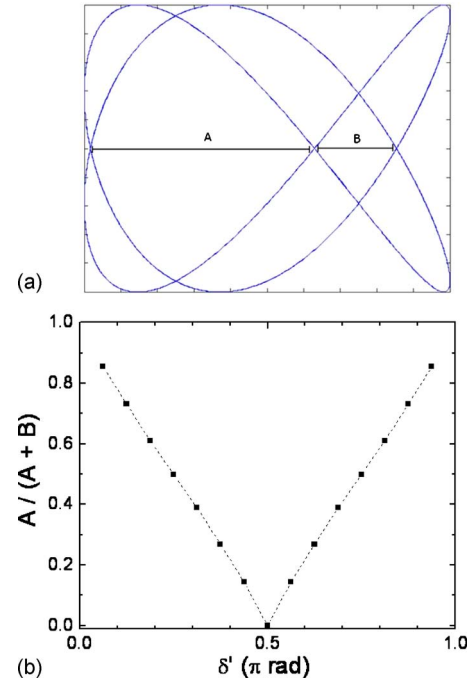


Fig. 4. (a) Relation between the effective phase δ_{eff} and the ratio A/B defined in (b) for the Lissajous figure corresponding to $K=3/2$.

To obtain quantitative information from these videos, we have to first find a way to relate the observed pictures to the value of δ_{eff} . The way in which we solved this problem is illustrated in Fig. 4. In Fig. 4(a), we define (for $K=3/2$) the parameters A and B . By performing simulations of the Lissajous figures at different phase factors, from $\delta_{\text{eff}}=0$ to $\delta_{\text{eff}}=\pi$, we can relate the ratio $A/(A+B)$ to the corresponding value of δ_{eff} , as shown in Fig. 4(b). In this way, every figure can be related to a value of δ_{eff} , which allows to plot δ_{eff} as a function of time. There is a limit to the resolution of the values of A and B , that is, to the smallest δ_{eff} value that can be distinguished. This number is determined mostly by the resolution of the video recorder and the thickness of the laser pointer used. In our case, we estimate this accuracy to be $\delta_{\text{eff}} \approx \pi/32$.

To confirm that the origin of the observed time drift comes from the fact that $\Delta\omega_2 \neq 0$, we performed several experiments in which the frequency of one of the tuning forks was slightly modified. This modification was done by varying the position of the rubber band of mass Δm , added to one of the arms of the tuning fork (see the inset in Fig. 2). Typical values were $\Delta m \approx 0.40$ g, and the positions were varied in the range of 5–8 cm from the end of the arms of the tuning fork. The results obtained for $K=3/2$ are shown in Fig. 5. We see that the effective phase factor strongly depends on the position of Δm with respect to the end of the arm. In particular, we see that T^* can be made larger or shorter with respect to the initial case $\Delta m=0$ (black dots, $T^*=2$ s). In this way, we were able to modify T^* by more than one order of magnitude.

Figure 5 shows our main result, namely, that the resonant frequencies of the two tuning forks can be matched very closely to the desired ratio ($K=3/2$ in this case) by looking at the corresponding Lissajous figures. The smaller the slope in this plot, the closer the two frequencies are to this ratio. In particular, we were able to tune these frequencies so that the

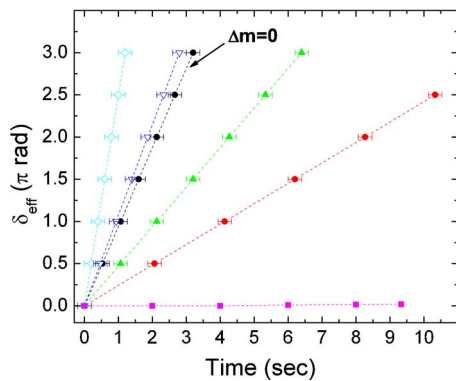


Fig. 5. Evolution of the effective phase δ_{eff} for $K=3/2$. The different cases correspond to different positions of a small mass Δm attached to one of the arms of the tuning fork, in the way shown in the inset of Fig. 2. The case corresponding to $\Delta m=0$ (black dots) is indicated by the arrow. The lines are a guide to the eye.

observed figures remained frozen during 140 frames, which corresponds to 9.3 s. If we consider a value for $\delta_{\text{eff}} \approx \pi/32$ (our estimated experimental accuracy, see above), it follows that these data points reach a value $\delta_{\text{eff}}=2\pi$ after a time $T^* = 9.3 \times 2 \times 32 = 595.2$ s. According to Eq. (12), it follows that the two frequencies differ by $\Delta\omega_2 = 0.010$ Hz. If we consider that the frequencies are on the order of 300 Hz, we conclude that our method is able to determine variations on the order of 0.01%.

The value of $\Delta\omega_2 = 0.010$ Hz is one order of magnitude less than the resolution of the direct method described in Sec. II, which was the one we used to determine the frequencies shown in Table II. For larger $\Delta\omega_2$ values, we have checked

that the same results are obtained with an analysis based on the Lissajous figures, using Eq. (2), and the direct frequency measurements described in Sec. II.

From Eq. (12), it is clear that to obtain a very small $\Delta\omega_2$, we need very large T^* values. As mentioned, we were able to keep a figure frozen for about 10 s. This limit is imposed by the mechanical attenuation of the vibrating tuning forks, and it is difficult to increase it significantly. There is still room for improvement in the resolution of δ_{eff} , which could be achieved by combining a thinner laser pointer with a higher resolution video recorder. In this way, it should be possible to obtain a resolution close to $\Delta\omega_2 \sim 0.001$ Hz.

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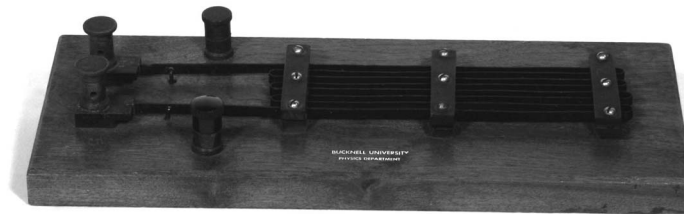
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⁶Crystals were float-zone grown n-Si(111) (phosphorus doping, with a resistivity of 20 Ω cm and low oxygen content of 3×10^{15} cm⁻³) and were double-sided polished with a native oxide surface. The crystals were provided by B. Surma (ITME, Warsaw).



Standard Resistance. This standard resistance came to the Greenslade Collection from Bucknell University, and may have been made in 1925 by two Bucknell undergraduates. It follows the standard design of Leeds & Northrup for 0.1 ohm and 0.01 ohm standard resistors, but has the unusual value of 0.2 ohm. The current connections are the two heavy binding posts on the left, designed to carry up to 6 A, and the lighter potential contacts are to their right. At one time a valuation of \$20 was painted onto the bottom of the base; the roughly-equivalent L&N instrument was about three times this. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College.)