

FEBRUARY 01 1989

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Phys. Teach. 27, 98–101 (1989)

<https://doi.org/10.1119/1.2342678>



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Three-Dimensional Lissajous Figures

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A mechanically driven device is described that can generate three-dimensional harmonic space figures with different frequencies and phase angles on the X-, Y-, and Z-axes. Stereoscopic photographs allow the user to view the three-dimensional figures.

Introduction

Lissajous figures are curves which represent the path of a particle that executes two simple harmonic motions at right angles to each other at the same time. Many current physics laboratory manuals use Lissajous patterns as a means of comparing frequencies with an oscilloscope. It has been noted that small frequency differences produce patterns that give the illusion of rotating three-dimensional figures.¹ Such rotating figures lead to the question, "What would Lissajous figures with three independent variables look like?" Two such devices which produce harmonic curves in space are described by Sutton.² These devices produce figures which are observed in real time, but they are limited in the variety of frequency and phase angle combinations that can be employed.

Description of the Apparatus

The three-dimensional figure is traced in space by the intersection of a vertical rod, which executes horizontal simple harmonic motion in the X and Y directions, with a laser-generated horizontal plane of light that oscillates with simple harmonic motion in the Z direction by use of a pendulum-like optical lever (see Fig. 1).

The harmonic oscillations of the apparatus are based on the circular motion of three crank handles driven by a single 12-V windshield wiper motor (see Fig. 2). Parts

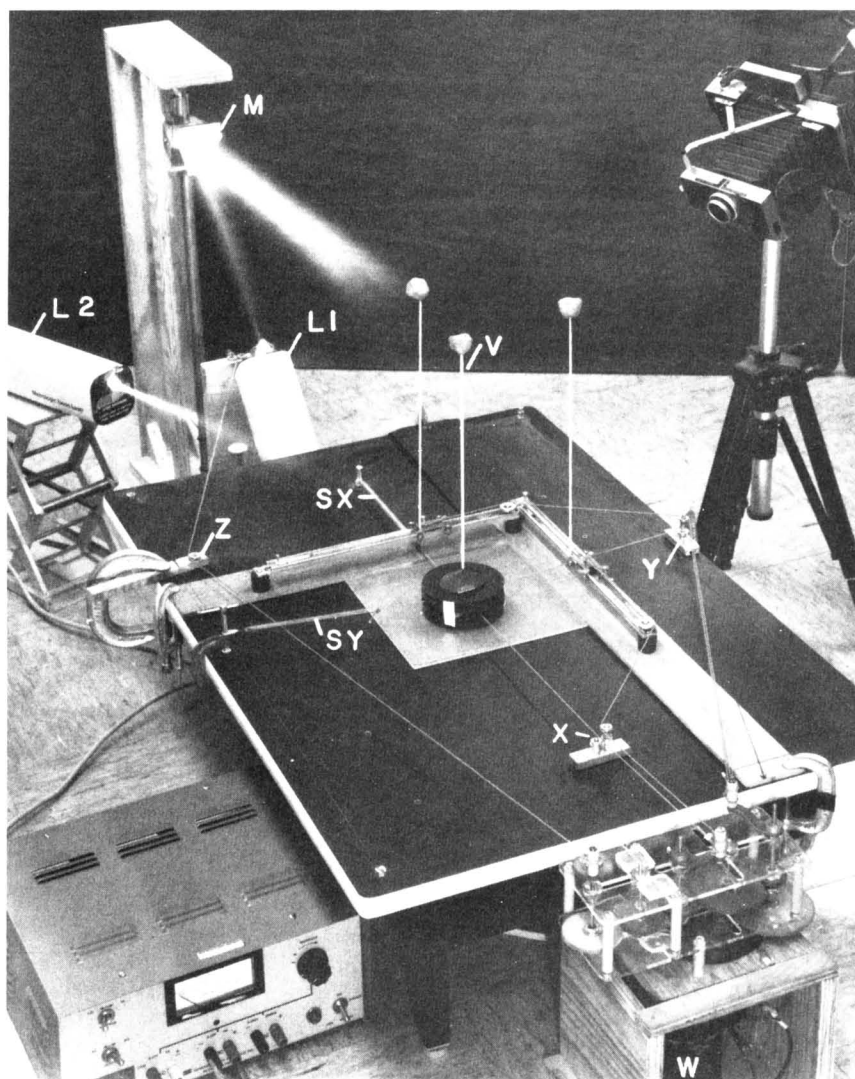


Fig. 1. Device to trace three-dimensional Lissajous patterns. W, windshield wiper motor; V, vertical X, Y, Z plotting rod; X, X-axis drive strings and pulleys; Y, Y-axis drive strings and pulleys; Z, Z-axis drive string and pulley; SX, X-axis tension spring; SY, Y-axis tension spring; L1, laser with cylindrical lens; M, front surface mirror; L2, stationary laser with cylindrical lens.

used to construct the gear train came from a mechanical demonstration set called "Compact Technik."³ A similar gear train could be designed and built from stock parts.⁴

The frequency of rotation of the crank handles

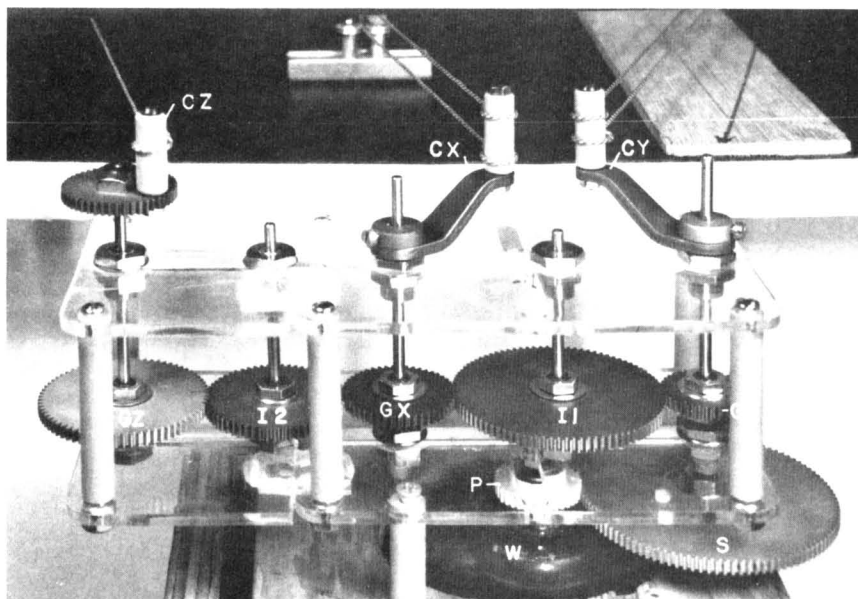


Fig. 2. Gear train. W, windshield wiper motor; P, 36-tooth pinion gear; S, 108-tooth spur gear; GY, 36-tooth Y-axis drive gear; I1, 96-tooth idler gear; GX, 48-tooth X-axis drive gear; I2, 60-tooth idler gear; GZ, 72-tooth Z-axis drive gear; CY, Y-axis crank; CX, X-axis crank; CZ, Z-axis crank handle.

depends on the number of teeth on the drive gears. The three drive gears have 36, 48, and 72 teeth, respectively, giving the crank handles a frequency ratio of 4:3:2. The drive gears alternate with idler gears to provide the necessary clearance for the crank handles to rotate freely. Medium-weight string connects the crank handles with the vertical X, Y, Z plotting rod. A 5.5-cm-long spring ($k = 10.7 \text{ N/m}$) secured to the free end of each drive string provides tension (see Fig. 1). For convenience the center crank is used to generate displacement in the X direction. Small ball-bearing pulleys, placed on either side of the X-axis drive-string, damp the unwanted component of the crank handle's rotation. For displacement along the Y-axis, a single ball-bearing pulley functions to damp the unwanted component of the crank handle's rotation and change the direction of the displacement so that it is perpendicular to the X-axis. Both the X and Y crank handles have a radius of 4 cm, thus the amplitude of oscillation along both the X- and Y-axes is 4 cm.

Where the X and Y drive strings cross, a vertical straight pin pushed through both strings connects them. The straight pin is anchored between two air table pucks with bushings cut from 1.58-mm-diameter telescoping brass tubing stock. The two pucks are separated with 1-cm-thick plastic spacers. The vertical plotting rod, made from 30-cm-long, 2.38-mm-diameter brass tubing, is anchored in the top puck and is painted white. To reduce friction and vibration, the bottom puck rests on a sheet of glass 20 cm $\times$ 20 cm.

The optical lever, constructed from a 50-cm length of 1.27-cm-diameter EMT steel conduit, is connected by an EMT coupling to a 1.27-cm-diameter wing top Bunsen

burner attachment (see Fig. 3). The wing top is crimped and bolted to a 2.54-cm steel hinge. The hinge is secured to a wooden frame so that the lower end of the conduit swings in one plane parallel to the X-axis. A 6 cm $\times$ 9 cm front surface mirror is mounted on a U-shaped bracket secured to an adjustable wooden block 9 cm below the hinge. The Z-axis drive string is connected through a series of pulleys to the conduit near the free end. A spring placed between the conduit and the wooden frame provides tension. A laser, aligned along the X-axis, is fitted with a 4-mm-diameter cylindrical lens cut from glass rod stock to create a horizontal plane of light. The laser and the mirror are adjusted so that the reflected horizontal plane of light intersects the vertical X, Y, Z plotting rod. The intersection of the vertical rod with the horizontal plane of light produces a point of light. When the

Z-axis crank rotates, the optical lever causes the horizontal plane of light to execute simple harmonic motion in the Z direction. The amplitude of oscillation along the Z-axis is set at approximately $\pm 4 \text{ cm}$ to match the amplitude of oscillation on the X- and Y-axes. This is accomplished by adjusting the variables of the optical lever. The combination used was determined experimentally. The Z-axis crank has a 1.2-cm radius. The Z-axis drive string attaches to the free end of the optical lever 60 cm below the hinge. The center of the mirror is 9 cm below the hinge, and the mirror is placed 85 cm from the X, Y, Z plotting rod. The combined effect of simple harmonic motions on the X-, Y-, and Z-axes is that a point of light traces out a three-dimensional figure in space.

An additional laser with a 4-mm-diameter cylindrical lens projects a fixed horizontal plane of light to record the X and Y pattern below the three-dimensional figure. Two additional vertical rods have been added to record the X and Z and the Y and Z patterns. These two rods are mounted on two slide wire and pen plate assemblies from a surplus, dual trace X, Y plotter.⁵ The two assemblies are mounted parallel to the X- and Y-axes, respectively, and are driven by additional drive strings. Balls of clay placed on top of each rod help damp small vibrations of the rods.

The stereo photographs were made with a single Polaroid camera using Polaroid 665 positive/negative film. A time exposure was taken as the apparatus completed its pattern of motion. A second exposure was taken after the camera was moved 5 cm to the right, perpendicular to the line of sight. After the negatives were processed they were trimmed to size and aligned.

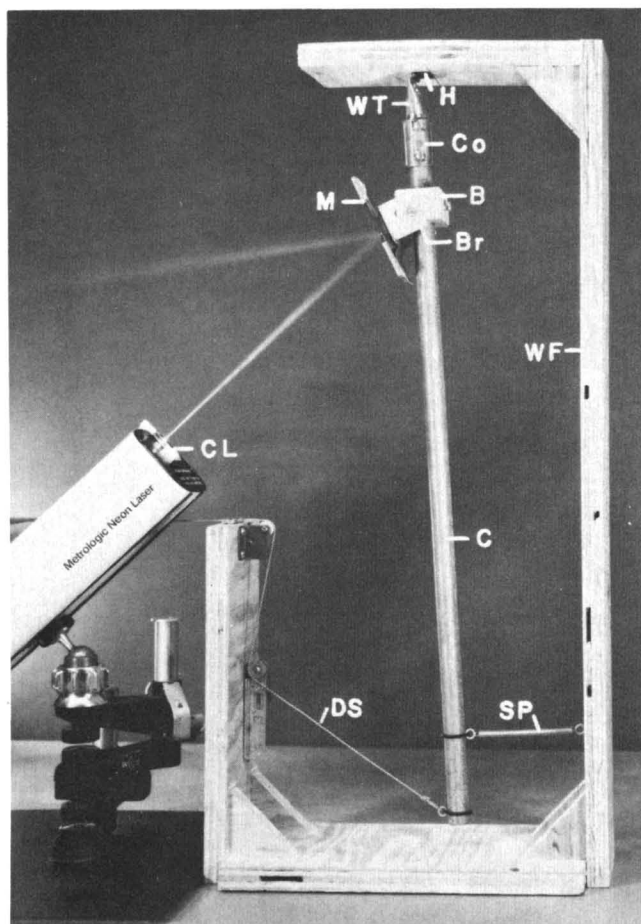


Fig. 3. Optical lever assembly. WF, wooden frame; H, hinge; WT, wing top; Co, EMT coupling; C, steel conduit; M, front surface mirror; Br, U-shaped bracket; B, wooden block; DS, drive string; SP, tension spring; CL, cylindrical lens.

The final stereo pairs were contact printed from these negatives.

Viewing the Stereo Pairs

Our binocular vision allows us to judge close distances very accurately because the visual cortex of the brain is able to superimpose and interpret the two images focused by the lenses of the eyes on the right and left retinas. Stereo photos present a slightly different image to each eye, thus providing enough information to visualize depth in a given scene. The stereo photos provided here can be viewed with a standard stereoscope used to view stereo aerial photos.⁶ Another method is to use two medium focal length convex lenses (fl 120 mm to 180 mm). Place the stereo pair flat on a table. Bring the lenses up to your eyes and adjust the distance between the photos and the lenses until the figures are in focus. Then, adjust the distance between the two lenses by moving them toward and away from the bridge of your nose while keeping them the same distance from the page. The two images will seem to snap together, creating a three-dimensional set of figures (see Figs. 4 and 5).

When viewing both Figs. 4 and 5, the uppermost central figure is the combined X, Y, Z trace. The straight line on the lower left is parallel to the X-axis and the line on the lower right is parallel to the Y-axis. The lower figure in the center illustrates displacement along the X- and Y-axes in a horizontal plane. The two other figures on the upper left and the upper right represent the combined displacement along the X- and Z-axes and the Y- and Z-axes, respectively. These two figures appear as two-dimensional patterns drawn on two vertical planes that are perpendicular to each other.

The Equations of Motion

Three-dimensional harmonic space figures can be described with the following equations:

$$\begin{aligned} X &= a_1 \sin(n_1 \omega t + \theta_1) \\ Y &= a_2 \sin(n_2 \omega t + \theta_2) \\ Z &= a_3 \sin(n_3 \omega t + \theta_3). \end{aligned} \quad (1)$$

Since the device is mechanical the origin of time, t , is the same for all three axes. The amplitude of the displacement has been adjusted to be equal on all three axes and may be taken to be one so that $a_1 = a_2 = a_3 = 1$. The initial phase angles $\theta_1, \theta_2, \theta_3$, can also be adjusted by setting the crank handles to the desired angle at $t = 0$. Simple and elegant figures are produced when $n_1/n_2, n_1/n_3$, and n_2/n_3 are ratios of small whole numbers. Simple frequency ratios are obtained by choosing gears with the appropriate number of teeth.

Figure 4 is described by the following equations:

$$\begin{aligned} X &= \sin(3\omega t + 0^\circ) \\ Y &= \sin(4\omega t + 0^\circ) \\ Z &= \sin(2\omega t + 0^\circ). \end{aligned} \quad (2)$$

These equations describe Fig. 5:

$$\begin{aligned} X &= \sin(3\omega t - 90^\circ) \\ Y &= \sin(4\omega t - 90^\circ) \\ Z &= \sin(2\omega t + 90^\circ). \end{aligned} \quad (3)$$

Using Three-Dimensional Space Figures in the Classroom

Frank G. Kariotis has described a method for plotting and analyzing Lissajous patterns using a sine-sine coordinate grid.⁷ After a discussion of how to use the grid, I gave the assignment to plot a series of two-dimensional patterns as homework before lab. Following the routine Lissajous oscilloscope lab, I proposed the question, "What would three-dimensional Lissajous figures be like?" The students were then asked to use scissors and tape to assemble a three-sided figure to illustrate the displacement of a point moving with simple harmonic motion along three perpendicular axes (Fig. 6). I challenged them to construct such a figure using lengths of solid-core copper wire. It did not take long for several students to construct the space figure pictured on the right in Fig. 6.

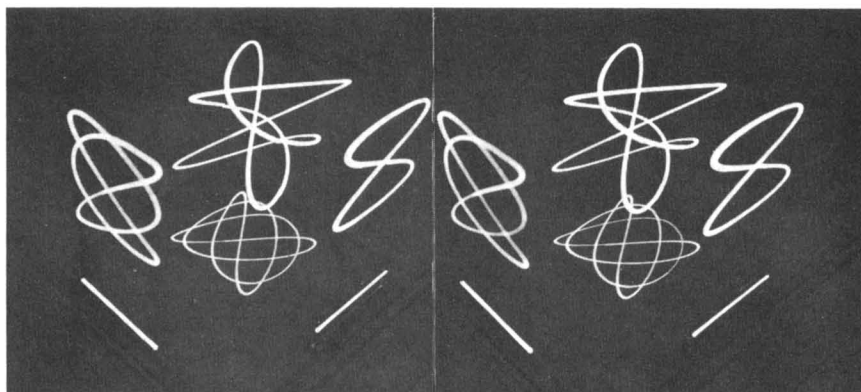


Fig. 4. Stereo photograph of two-dimensional Lissajous patterns in three perpendicular planes and a three-dimensional composite figure. Left figure X, Z ratio 3:2, right Y, Z ratio 4:2 (2:1), and lower center X, Y ratio 3:4. Upper center is the composite three-dimensional figure X, Y, Z, ratio 3:4:2. The initial phase angle for all three axes is 0° .

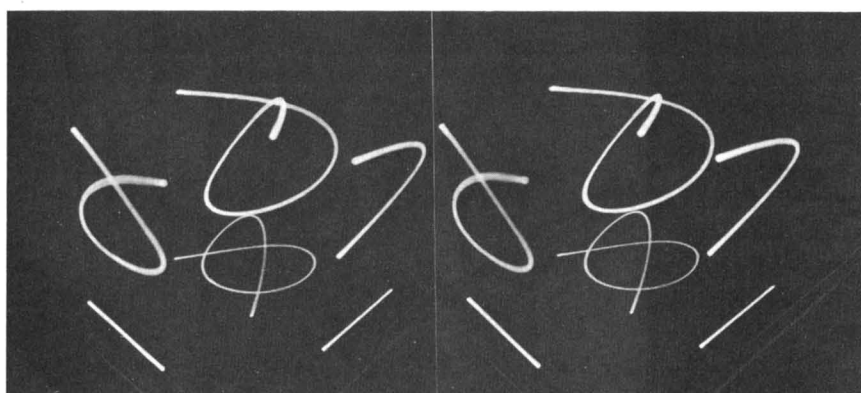


Fig. 5. Stereo photograph of three-dimensional Lissajous patterns. The frequency ratios are the same as in Fig. 4, but the initial phase angles have been set to -90° , -90° , and $+90^\circ$ for the X, Y, and Z axes, respectively.

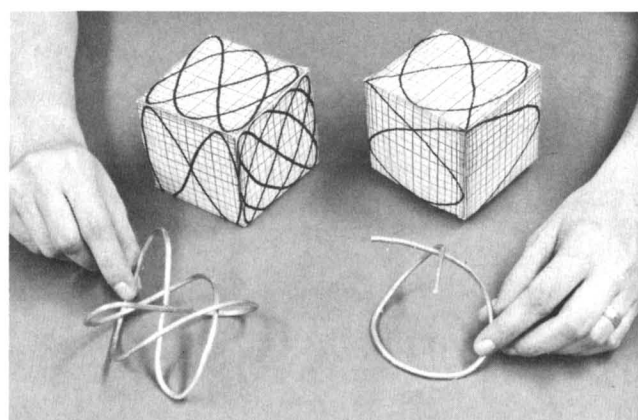


Fig. 6. Sine-sine grids cut and taped into three-sided figures and corresponding wire approximations of three-dimensional harmonic space figures.

The task to visualize the way that the three-plane figures, pictured on the left in Fig. 6, blend into one three-dimensional figure was much more difficult. This led to a discussion of how such a figure could be illustrated. The most frequent solution suggested required a formidable computer and sophisticated graphics software to plot and rotate the harmonic space figure. At this point I introduced the ignoble machine and demonstrated its use. The students watched the spot of light as it traced its sinuous path and observed how the pattern repeated as the crank handles went in and out of phase. Using stereoscopes, the students could readily visualize the harmonic space figures and the more persistent ones were able to construct reasonable approximations of the figures by bending copper wire (see Fig. 6). □

All photos taken by author.

References

1. Frank G. Karioris, *Phys. Teach.* **13**, 294 (1975).
2. Richard M. Sutton (Editor), *Demonstration Experiments in Physics*, (McGraw-Hill Book Company, Inc., New York, 1938), pp. 148-149.
3. Compact Technik sets were produced by Automat-Service of Schriesheim, West Germany and distributed by Stock Model Parts of New Hyde Park, New York. I have not been able to contact either company and do not know if these sets are currently available in the United States.
4. A good source of gears, shafts, shaft collars, bearings, pulleys, and tubing is Small Parts Inc., 6901 N. E. Third Ave., P.O. Box 3811736, Miami, FL 33238-1736.
5. These parts were taken from an obsolete Hewlett Packard Model 2D-2, X-Y recorder. Part nos. M-15152 and M-15217. They are not currently available from the manufacturer. An alternative sliding mechanism might be made from a length of HO railroad track with the plotting rod mounted on a model train wheel carriage or truck.
6. Student stereoscopes are available from Frey Scientific, 905 Hickory Lane, Mansfield, OH 44905. 1987-88 Catalog, Page 215, item #5264, \$2.95 ea.
7. Karioris, p. 294.