16. Twisted spaghetti

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Start with the assignment

16. Twisted spaghetti
When a bundle of spaghetti is twisted,
it might withstand higher transverse (side) forces than a straight, untwisted bundle.
Investigate the response of a twisted bundle to transverse stress
and identify the optimal twist that maximises tolerance to transverse stress.

Czech translation (we put some effort into it, might help understanding)

16. Zkroucené špagety
Když zkroutíme svazek špaget,
může vydržet působení větších příčných sil (působících z boku) než rovný, nezkroucený svazek.
Prozkoumejte odezvu zkrouceného svazku na příčné napětí
a najděte optimální zkroucení pro dosažení nejvyšší meze pevnosti při příčném zatížení. .

Clear structure in two sentences, each with two clauses

1. Problem definition
    a. Setup
    b. Phenomenon
2. Assignment—characteristically open
    a. Investigation
    b. Optimization

Start with resources at https://www.tmfcr.cz/

esp. the kit there https://kit.ilyam.org/Draft_2026_IYPT_Reference_kit.pdf

Typical structure

1. Illustrative picture

2. Video: https://www.youtube.com/watch?v=RwtXVW0IWEk

Try it   Simple enough to be shown at the tournament itself

connection to a problem from last year
there breaking, now adding a twist

3. General references, e.g. Wikipedia, for general concepts,
here https://en.wikipedia.org/wiki/Torsion_(mechanics)

4. Specific references—learn what is known

Acknowledge all references in the presentation to give credit and to show you are aware of results
Some publicly available, for others you need help from somebody with access
Material in the kit is typically in English, and so I use English here too

Basic concepts and phenomena: relevant, not everything you know

(as they sometimes encourage in school, at least did in my school)

General concepts:

- In this problem the basic concept is elasticity and strength
- Applied stress gives rise to strain, i.e., relative change of length
- For small enough stress and strain, they are proportional to each other—Hooke's law
- The proportionality constant is an elastic modulus in Pa

specific situation will give specific form and meaning

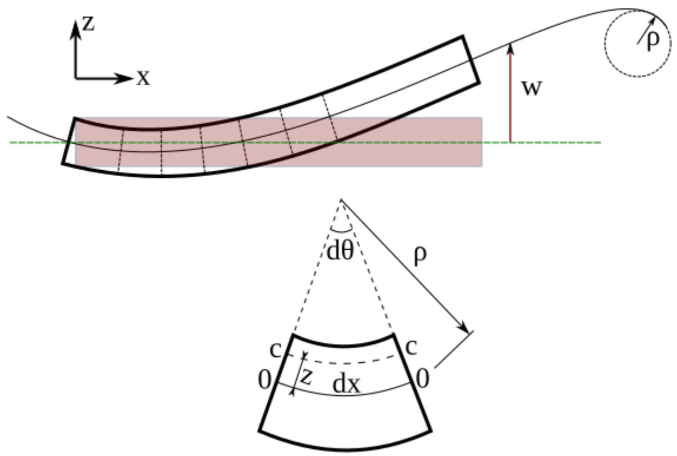
A bit more specific

Spaghetti has shape with very different dimensions: thin and long

The video first discusses bending

Deformation sideways from the long direction—stretching and compressing strain

I will show the main steps in the derivation of the beam equation since they contain generally important ingredients



- length is given by the distance to the center and by the angle
- there is the unstrained neutral axis

with z measured from the neutral axis, the strain is $$\frac{(z+\rho)d\theta - \rho d\theta}{\rho d\theta} = \frac{z}{\rho}$$

we see subtraction of the neutral length in the numerator and cancellation of the angle in the fraction

the proportionality constant in Hooke's law is the Young's modulus E so the stress is $$E\frac{z}{\rho}$$

Acknowledging the source

https://en.wikipedia.org/wiki/Euler%E2%80%93Bernoulli_beam_theory

Force obtained from stress by multiplication by area

force along the beam direction, which is x here $$d\vec{F} = E\frac{z}{\rho}dA\vec{e}_x$$

symbols d here mean that we are taking

a small area dA from the cross section giving us a small contribution to the force $d\vec{F}$

Small contribution to the torque $$d\vec{M} = -z\vec{e}_z \times d\vec{F} = -\vec{e}_y E\frac{z^2}{\rho}dA$$ minus sign for the right-hand rule

The total torque

$$\vec{M} = \int d\vec{M} = -\vec{e}_y\frac{E}{\rho}\int z^2 dA = -\vec{e}_y\frac{EI}{\rho} = -\vec{e}_y EI\frac{d^2w}{dx^2}$$

The steps involve integration, introduction of the second moment of area I, and of the curvature $$\kappa = \frac{1}{\rho} = \frac{d^2w}{dx^2}$$

with differentiation

Shear force Q satisfies $dM = Qdx$ Load q to keep equilibrium satisfies $dQ = -qdx$

so $q = -\frac{d^2M}{dx^2}$

Dynamics: the force gives acceleration by Newton's second law of motion

$$EI\frac{\partial^4 w}{\partial x^4} + \mu\frac{\partial^2 w}{\partial t^2} = 0$$

where $\mu = \rho A$ is the linear mass density in $\mathrm{kg/m}$

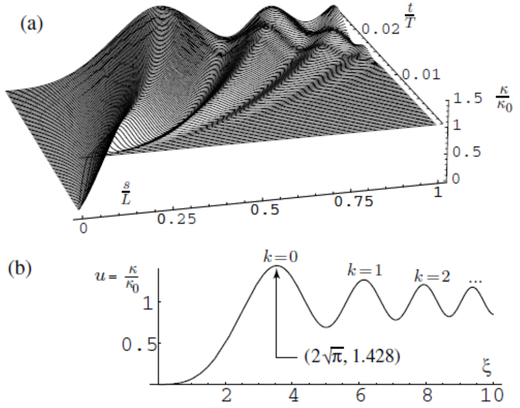
Same equation for the curvature $EI\frac{\partial^4\kappa}{\partial x^4}+\mu\frac{\partial^2\kappa}{\partial t^2}=0$

We see the appearance of the ratio $\frac{EI}{\mu}$ with units or dimension $\frac{\mathrm{Pa\,m^4}}{\mathrm{kg\,m^{-1}}}=\frac{\mathrm{N\,m^{-2}m^4}}{\mathrm{kg\,m^{-1}}}=\frac{\mathrm{kg\,m\,s^{-2}m^{-2}m^4}}{\mathrm{kg\,m^{-1}}}=\frac{\mathrm{m^4}}{\mathrm{s^2}}$

We define $\gamma=\sqrt{\frac{EI}{\mu}}$ with units $\frac{\mathrm{m^2}}{\mathrm{s}}$ like the diffusion constant

This suggests self-similarity, that is, dependence of the solution on just one dimensionless variable $\xi=\frac{x}{\sqrt{\gamma t}}$

The equation $\frac{\partial^4\kappa}{\partial x^4}+\frac{1}{\gamma^2}\frac{\partial^2\kappa}{\partial t^2}=0$ becomes $4u''''(\xi)+\xi^2u''(\xi)+3\xi u'(\xi)=0$ $\kappa(x,t)=\kappa_0 u(\xi)$



with the solution $u(\xi)=2S\left(\frac{\xi}{\sqrt{2\pi}}\right)$ where $S(x)=\int_0^x dy\,\sin\left(\frac{\pi}{2}y^2\right)$

is the Fresnel integral (sometimes defined without $\frac{\pi}{2}$)

known from Fresnel near-field diffraction

and related to the error function

Due to the analogy with diffraction, curvature increases in several places and the spaghetti breaks

B. Audoly and S. Neukirch. Fragmentation of rods by cascading cracks: Why spaghetti does not break in half. Phys. Rev. Lett. 95, 9, 095505 (2005)

Y. Zhang, X. Li, Y. Dai, and B. Sun. Spaghetti breaking dynamics. Preprints 2021030311 (2021)

Nature uses only the longest threads to weave her patterns,
so that each small piece of her fabric reveals the organization of the entire tapestry.

Richard Feynman

A lot of times he'd get crazy ideas of how physical things worked. He was always putting his theories to the test, and that was a great thing about Richard—whenever you asked a question and couldn't think of the answer, Richard would say, "Well, what experiment can we do to figure it out?"

Once we were making spaghetti, which was our favourite thing to eat together. Nobody else seemed to like it. Anyway, if you get a spaghetti stick and you break it, it turns out that instead of breaking in half, it will almost always break into three pieces. Why is this true—why does it break into three pieces? We spent the next two hours coming up with crazy theories. We thought up experiments, like breaking it under water because we thought that might dampen the sound, the vibrations. Well, we ended up at the end of a couple of hours with broken spaghetti all over the kitchen and no real good theory about why spaghetti breaks in three. A lot of fun, but I could have blackmailed him with some of his spaghetti theories, which turned out to be dead wrong!

Christopher Sykes

The long thread in Feynman's spaghetti problem appears to be the Fresnel integral
connecting spaghetti to near-field diffraction and the error function

The introductory video then discusses the new twist on the spaghetti story

Torsion

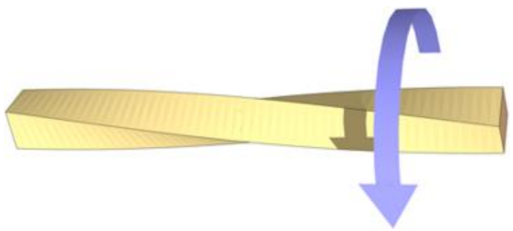
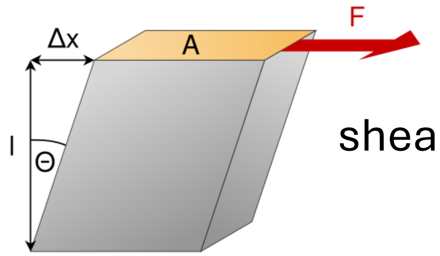


figure from the Wikipedia page in the kit

Deformation around the long direction—shear stress and strain rather than extension or compression



shear stress $\tau_{xy} = \frac{F}{A}$ and shear strain $\gamma_{xy} = \frac{\Delta x}{l}$ are proportional via the shear modulus G

$$\tau_{xy} = G\gamma_{xy}$$

https://en.wikipedia.org/wiki/Shear_modulus

Dependence of the stress, strain, and modulus on the direction reveals the tensor nature of these variables

you can get here a feel for tensors; in fact, they come from elasticity, hence tensors

Derivation of the twist from shear similar to derivation of bending from extension and shrinking, with some differences

Briefly,
along the circle

$$\Delta x = r\varphi$$

$$A = 2\pi r dr$$

$$dF = G\frac{r\varphi}{l}2\pi r dr$$

$$dT = r dF = G\frac{\varphi}{l}r^2 2\pi r dr$$

$$r^2 2\pi r dr = dI_{zz}$$

$$I_{zz} = \int dI_{zz} = 2\pi \int_0^r r^3 dr = \frac{\pi r^4}{2}$$

$$T = G\frac{\varphi}{l}I_{zz}$$

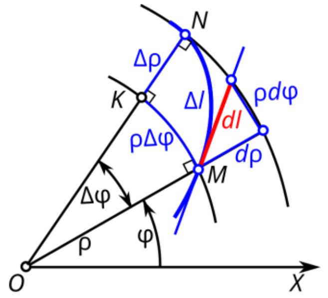
Twist angle proportional to length and to the torque

z axis now along the beam—beware of different conventions in different sources

Next step from a single twisted spaghetto to a twisted bundle

Polar coordinates in the plane perpendicular to the bundle $(dl)^2 = (d\rho)^2 + \rho^2(d\varphi)^2$

Pythagorean theorem



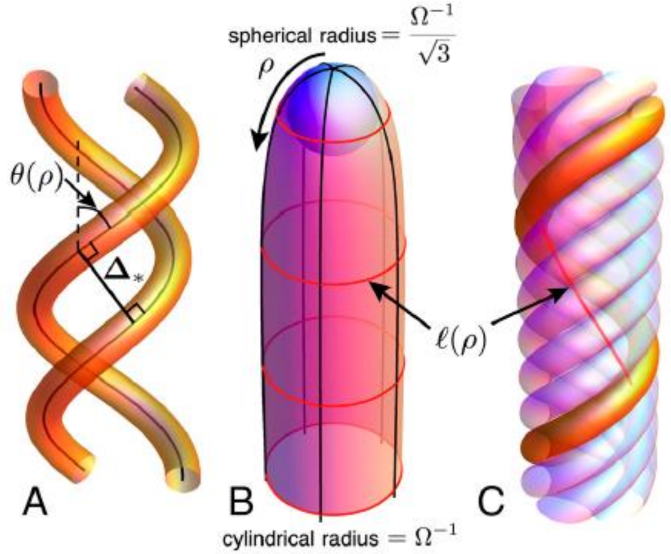
https://commons.wikimedia.org/wiki/File:Line_element_in_polar_coordinates.svg

Twist gives also a rise with the rate of helical rotation $\Omega = 2\pi/P$ where P is the pitch, the height of a single turn

$$\Delta^2(z) = (d\rho)^2 + \rho^2(d\varphi + \Omega z)^2 + z^2$$

We are looking for z_* minimizing $\Delta^2(z)$, see figure A

$$\Delta^2(z_*) \equiv (ds)^2 = (d\rho)^2 + \rho^2(d\varphi)^2 \frac{1}{1 + (\Omega\rho)^2}$$



due to the appearance of the denominator $1 + (\Omega\rho)^2$

we have effectively curved, or non-Euclidean, asymptotically flat surface

R. Bruss and G. M. Grason. Non-Euclidean geometry of twisted filament bundle packing. Proc. Natl. Acad. Sci. U.S.A. 109, 27, 10781-10786 (2012)

Without the twist, the plane is flat and can be filled by the sixfold close-packed arrangement, but the curved surface necessarily has defects. The characteristic

$$Q = \sum_n (6-n)V_n$$

is topological

In the limit of an infinite bundle can be calculated from the Gaussian curvature $K_G = \kappa_{min}\kappa_{max}$

product of the two extreme 1d curvatures we met in bending

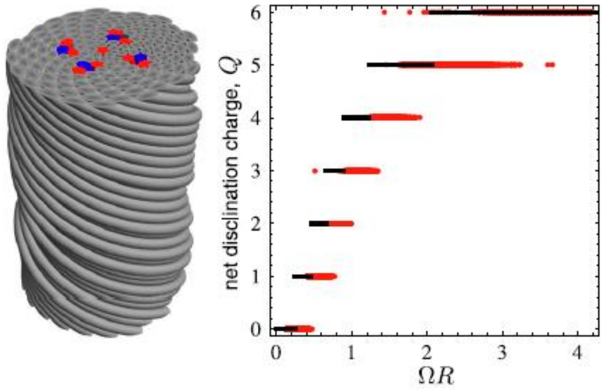
using the Gauss-Bonnet theorem $Q = 6\chi$ where $\chi = \frac{1}{2\pi}\int dAK_G$ is the Euler characteristic

The surface has a positive Gaussian curvature due to the twist

$$K_G = \kappa_{min}\kappa_{max} = \frac{3\Omega^2}{(1+(\Omega\rho)^2)^2}$$

for which $\chi = 1$ so $Q = 6$

Indeed



Twisted spaghetti give us various Feynman long threads of mathematical structure, such as:

- integration
- differentiation
- dimensional analysis
- self-similarity
- Fresnel integral
- tensors
- non-Euclidean geometry
- topology