Optical wave propagation in active media: gyrotropic–gyrochiral media

Habib Taouk

University of Pittsburgh, 504 East Main Street, Titusville, Pennsylvania 16354

Received May 10, 1996; revised manuscript received January 2, 1997; accepted January 28, 1997

A comprehensive study of electromagnetic wave propagation is conducted in linear, homogeneous, and lossless biuniaxial media that are optically active and placed in a constant magnetic field. The dispersion equation is derived, and the coupled effects of the optical activity and Faraday effect are discussed. These effects include the inequality of the refractive indices in opposite directions of propagation and the conceptual possibility of trirefringence. Clear manifestations of these effects occur when the direction of propagation is parallel to that of the constant magnetic induction field. In such a case, closed-form solutions for the refractive indices, the field vectors, and the wave polarizations are derived. © 1997 Optical Society of America
[S0740-3232(97)00808-9]

1. INTRODUCTION

Optical activity has been a topic of interest since the early 19th century. Excellent reviews of recent contributions to this field can be found in the literature.[1,2] With the prospects of optical chips taking over the process of computations at the speed of light, optically active media may prove to be useful in meeting the hardware requirements. The use of chiral materials has been investigated in optical fibers and waveguides[3–8] and as polarization transformers,[9] light modulators,[10,11] and optically unidirectional isolators.[12]

Although the majority of the naturally occurring active media are anisotropic,[13] most of the attention has been given to isotropic chiral media. The characteristics of wave propagation in optically active anisotropic media were not investigated until recently.[14–20] In addition, except for few recent papers,[14,20–25] the lack of studies in media that possess natural and induced optical activities simultaneously has masked interesting results that may well be needed for the advance of optical technology.

This paper is a comprehensive coordinate-free study of electromagnetic wave propagations in biuniaxial, homogeneous, linear, and lossless media that are optically active and placed in a constant magnetic induction field, $\mathbf{B}_0 = \mathbf{B}_0\hat{\mathbf{c}}$, that is applied in the direction of the optic axis of the material, $\hat{\mathbf{c}}$.

We denote a scalar by a light face letter (e.g., k, μ), a vector by boldface letter (e.g., $\mathbf{k}$, $\mathbf{H}$), a unit vector by a caret above the vector (e.g., $\hat{\mathbf{e}}$), a matrix by an overbar (e.g., $\bar{\epsilon}$), the transpose of a matrix by a tilde (e.g., $\tilde{\bar{\gamma}}$), a Hermitian conjugate by a dagger (e.g., $\bar{\epsilon}^{\dagger}$), and a complex conjugate by an asterisk (e.g., $\mathbf{k}^*$, $\bar{\epsilon}^*$).

2. BASIC EQUATIONS

The study of electromagnetic phenomena involves the solution of Maxwell's equations (ME's) in conjunction with the constitutive relations (CR's).

A. Maxwell's Equations

We consider monochromatic simple-harmonic time-varying fields. In this case the ME's take the following form in terms of the complex-constant amplitudes of the field vectors[26]:

$$\mathbf{k} \times \mathbf{E} = \omega\mathbf{B}, \tag{1a}$$

$$\mathbf{k} \times \mathbf{H} = -\omega\mathbf{D}, \tag{1b}$$

$$\mathbf{k} \cdot \mathbf{D} = \mathbf{k} \cdot \mathbf{B} = 0, \tag{1c}$$

where $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$ represent the complex-constant amplitudes of the electric and magnetic field intensities and the electric and magnetic flux densities, respectively. $\mathbf{k} = k\hat{\mathbf{k}}$ is the wave vector, $k = 2\pi/\lambda$ is the wave number, $\hat{\mathbf{k}}$ is the wave normal, which is a unit vector in the direction of $\mathbf{k}$, λ is the wavelength, and ω is the angular frequency. The plane wave is perpendicular to $\mathbf{k}$, and it travels with the phase velocity $\mathbf{v}_p = v_p\hat{\mathbf{k}} = (\omega/k)\hat{\mathbf{k}}$. We define the refractive-index vector by

$$n = \mathbf{k}/k_0 = (c/\omega)\mathbf{k} = (k/k_0)\hat{\mathbf{k}}, \tag{2}$$

where $c = (\epsilon_0/\mu_0)^{-1/2} \approx 3 \times 10^8$ m/s is the velocity of light in free space, $k_0 = \omega(\epsilon_0\mu_0)^{1/2} = \omega/c$ m^{-1} is the wave number in free space, and $\epsilon_0 \approx 8.854 \times 10^{-12} \approx (1/36\pi) \times 10^{-9}$ F/m and $\mu_0 = 4\pi \times 10^{-7}$ H/m are the permittivity and the permeability of vacuum, respectively.

B. Constitutive Relations

Electromagnetic properties are specified by their CR's. In a linear medium the field vectors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$ are linearly related. The derivation and justification of the CR's in optically active anisotropic media were the main topic of several papers.[27–32] However, there is little agreement on the form of the CR's that best describe these media. We can select an appropriate set by following a few established guidelines.

Nonsymmetric sets of the CR's, such as that derived by Born,[28] are not considered because they have been shown

to characterize nonreciprocal media.[33–35] The controversy over which field vectors should be paired has existed for a long time.[36–39] We conclude that it is physically more meaningful to pair (**E**, **B**) and (**D**, **H**) but it is more convenient to pair the mathematically analogous vectors (**E**, **H**) and (**D**, **B**). I chose the latter set of pairs since we can determine **H** and **E** experimentally inside a material.[40] In addition, Silverman[33] suggested that the fundamental forms of physically acceptable constitutive relations should be invariant, as in ME's, under the duality transformations of fields.[41] Following this guideline, it would be impossible to establish duality transformations for the paired fields (**E**, **B**) and (**D**, **H**) in contrast to (**E**, **H**) and (**D**, **B**). Consequently, we adopt the following set of CR's in lossless media[26]:

$$\mathbf{D} = \bar{\epsilon} \cdot \mathbf{E} + \bar{\xi} \cdot \mathbf{H}, \tag{3a}$$

$$\mathbf{B} = \bar{\xi}^{\dagger} \cdot \mathbf{E} + \bar{\mu} \cdot \mathbf{H}, \tag{3b}$$

where $\bar{\epsilon} = \bar{\epsilon}^{\dagger}$ and $\bar{\mu} = \bar{\mu}^{\dagger}$ are Hermitian tensors. Note that $\bar{\xi}$ is not invariant under a spatial inversion[41] and, hence, is a pseudotensor that can occur only in a medium without a center of symmetry.

The properties of various electromagnetic effects in nonconductive media can be found in the literature,[28,30,42] and they are summarized with the time-frequency transformations in Table 1. In a stationary medium $\mathrm{Re}(\bar{\xi}) = \bar{\mathbf{0}}$, and hence $\bar{\xi} = i\bar{\gamma}$, where $\bar{\gamma}$, the optical activity tensor, is real. Rewriting the CR's in terms of the relative dielectric, permeability, and optical activity tensors, we obtain

$$\mathbf{D} = \epsilon_0 \bar{\epsilon} \cdot \mathbf{E} + ic^{-1}\bar{\gamma} \cdot \mathbf{H}, \tag{4a}$$

$$\mathbf{B} = -ic^{-1}\bar{\gamma} \cdot \mathbf{E} + \mu_0 \bar{\mu} \cdot \mathbf{H}. \tag{4b}$$

C. Gyrotropic–Gyrochiral Media

Gyrotropic–gyrochiral media are characterized by the chirality, permittivity (dielectric), and permeability tensors

$$\bar{\gamma} = \begin{bmatrix} \gamma_1 & \gamma_x & 0 \\ -\gamma_x & \gamma_1 & 0 \\ 0 & 0 & \gamma_2 \end{bmatrix}, \tag{5}$$

$$\bar{\epsilon} = \begin{bmatrix} \epsilon_1 & i\epsilon_x & 0 \\ -i\epsilon_x & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{bmatrix}, \qquad \bar{\mu} = \begin{bmatrix} \mu_1 & i\mu_x & 0 \\ -i\mu_x & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{bmatrix}. \tag{6}$$

The tensor $\bar{\gamma}$ describes the optical activity of several types of crystal[36,43,44] that include biuniaxial crystals [monoclinic (class *2*) and orthorhombic (classes *222* and *mm2*)], uniaxial crystals [tetragonal (classes *4* and *422*), hexagonal and triangular (classes *3*, *32*, *4*, *422*, *6*, and *622*)], and isotropic [cubic (classes *432* and *23*)]. It has been shown[45,46] that the maximum value of the chirality coefficient γ in an isotropic medium is $\sqrt{\epsilon\mu}$.

Table 1. Properties of Various Electromagnetic Effects in Nonconductive Media

Type of Effect	Not Necessarily Dispersive	Essentially Dispersive
Reciprocal	Linear dielectric and magnetic birefringence $\mathrm{Re}(\epsilon, \mu) \rightarrow \mathrm{Re}(\epsilon, \mu)$	Natural optical activity $\mathrm{Im}(\xi) \rightarrow \mathrm{Im}(\xi)$
Nonreciprocal	Fresnel–Fizeau effect in moving media $\mathrm{Re}(\xi) \rightarrow -\mathrm{Re}(\xi)$	Dielectric and magnetic Faraday effects $\mathrm{Im}(\epsilon, \mu) \rightarrow -\mathrm{Im}(\epsilon, \mu)$

ϵ_x and μ_x are the contributions of the Faraday effect[26] that occurs in a ferrite or when a constant magnetic field is applied to isotropic materials or along the optic axis $\hat{c}$ of a uniaxial material such as rubidium nickel fluoride and ytterbium orthoferrite.[47]

A gyrotropic–gyrochiral medium provides cross coupling between the electric and the magnetic fields and gives rise to special effects that would not appear in the presence of either the chirality or the Faraday effect alone. It may occur in ferromagnetic materials such as gallium iron oxide in antiferromagnetic materials such as chromium oxide, or in spontaneously magnetized materials without application of an external biasing magnetic field, such as a ferrite.

3. HOMOGENEOUS EQUATIONS

ME's are sets of coupled equations that can be manipulated, together with the CR's to yield equations in which each of the field vectors **E**, **D**, **B**, and **H** is satisfied separately. In terms of the antisymmetric matrix $(\mathbf{k} \times \bar{\mathbf{I}})$, where $\bar{\mathbf{I}}$ is a 3×3 identity matrix, we can rewrite the ME's as

$$(\mathbf{k} \times \bar{\mathbf{I}}) \cdot \mathbf{E} = \omega \mathbf{B}, \tag{7a}$$

$$(\mathbf{k} \times \bar{\mathbf{I}}) \cdot \mathbf{H} = -\omega \mathbf{D}, \tag{7b}$$

$$\mathbf{k} \cdot \mathbf{D} = \mathbf{k} \cdot \mathbf{B} = 0. \tag{7c}$$

ME's in conjunction with the CR's lead to the following homogeneous equations:

$$(\bar{\mathbf{I}} + \bar{\mathbf{M}} \cdot \bar{\mathbf{N}}) \cdot \mathbf{E} = \mathbf{0}, \tag{8a}$$

$$(\bar{\mathbf{I}} + \bar{\mathbf{N}} \cdot \bar{\mathbf{M}}) \cdot \mathbf{H} = \mathbf{0}, \tag{8b}$$

where $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$ are given by

$$\bar{\mathbf{M}} = (\epsilon_0 \bar{\epsilon})^{-1} \cdot [ic^{-1}\bar{\gamma} + (\mathbf{k} \times \bar{\mathbf{I}})/\omega], \tag{9a}$$

$$\bar{\mathbf{N}} = (\mu_0 \bar{\mu})^{-1} \cdot [ic^{-1}\tilde{\bar{\gamma}} + (\mathbf{k} \times \bar{\mathbf{I}})/\omega]. \tag{9b}$$

Equation (9b) was obtained from Eq. (9a) by application of the following duality transformations:

$$\mathbf{E} \rightarrow \mathbf{H}', \qquad \mathbf{D} \rightarrow \mathbf{B}', \qquad \bar{\epsilon} \leftrightarrow \bar{\mu}, \qquad \bar{\gamma} \leftrightarrow \tilde{\bar{\gamma}}, \tag{10a}$$

$$\mathbf{H} \rightarrow -\mathbf{E}', \qquad \mathbf{B} \rightarrow -\mathbf{D}', \qquad \epsilon_0 \leftrightarrow \mu_0,$$

$$c, \ \omega, \ \mathbf{k} \leftrightarrow c, \ \omega, \ \mathbf{k}, \tag{10b}$$

under which the ME's and the CR's remain invariant. Equations (8) are valid if $|\bar{\epsilon}| \neq 0$ ($\pm\epsilon_x \neq \epsilon_1$) and $|\bar{\mu}| \neq 0$ ($\pm\mu_x \neq \mu_1$).

If only one determinant is zero, say, $|\bar{\epsilon}| = 0$, then Eq. (8b) cannot be given in a simple form but Eq. (8a) can be rewritten in the following form:

$$\{\epsilon_0\bar{\epsilon} + [ic^{-1}\bar{\gamma} + (\mathbf{k}\times\bar{\mathbf{I}})/\omega]\cdot(\mu_0\bar{\mu})^{-1}\cdot[ic^{-1}\tilde{\bar{\gamma}} + (\mathbf{k}\times\bar{\mathbf{I}})/\omega]\}y7\cdot\mathbf{E} = \mathbf{0}. \quad (11)$$

If $|\bar{\epsilon}| = 0$ and $|\bar{\mu}| = 0$ simultaneously, then the homogeneous equations can be derived only if

$$|ic^{-1}\bar{\gamma} + (\mathbf{k}\times\bar{\mathbf{I}})/\omega| \neq 0.$$

4. DISPERSION EQUATION

The dispersion equation characterizes the behavior of a plane wave in a medium. It relates the components of wave vector $\mathbf{k}$ to the dielectric, permeability, and chirality coefficients and to the angular frequency ω of the wave. It is also the equation of the wave vector surface that is mapped by the tip of $\mathbf{k}$ as it changes its direction in space.

A nontrivial solution for the homogeneous equations [e.g., Eq. (8a)] exists only if $|\bar{\mathbf{I}} + \bar{\mathbf{M}}\cdot\bar{\mathbf{N}}| = 0$. This determinant must be real because $(\bar{\epsilon}\cdot\bar{\mathbf{M}}\cdot\bar{\mathbf{N}})$ and $\bar{\epsilon}$ are Hermitian tensors, and it yields

$$|\bar{\mathbf{I}} + \bar{\mathbf{M}}\cdot\bar{\mathbf{N}}| = 1 + (\bar{\mathbf{M}}\cdot\bar{\mathbf{N}})_t + [\mathrm{adj}(\bar{\mathbf{M}}\cdot\bar{\mathbf{N}})_t + |\bar{\mathbf{M}}\cdot\bar{\mathbf{N}}| = 0. \quad (12)$$

Substituting Eqs. (5)–(7) into Eq. (12) and rearranging the terms in the order of the power of the refractive index $n = k/k_0$, we obtain

$$D_4 n^4 + D_3 n^3 + D_2 n^2 + D_1 n + D_0 = 0, \quad (13)$$

where

$$\begin{aligned}
D_4 &= (\epsilon_2\mu_2 - \gamma_2^2)(\cos\alpha)^4 + (\epsilon_1\mu_1 - \gamma_1^2)(\sin\alpha)^4 \\
&\quad + (\epsilon_1\mu_2 + \epsilon_2\mu_1 - 2\gamma_1\gamma_2)(\cos\alpha\,\sin\alpha)^2, \\
D_3 &= 2[\gamma_1(\epsilon_2\mu_x + \epsilon_x\mu_2) - \gamma_2(\epsilon_1\mu_x + \epsilon_x\mu_1)] \\
&\quad \times \cos\alpha(\sin\alpha)^2, \\
D_2 &= -2(\epsilon_2\mu_2 - \gamma_2^2)[(\gamma_1^2 - \gamma_x^2) + (\epsilon_1\mu_1 + \epsilon_x\mu_x)] \\
&\quad \times (\cos\alpha)^2 + [-2\gamma_1\gamma_2(\epsilon_1\mu_1 + \epsilon_x\mu_x) + (\epsilon_1\mu_2 \\
&\quad + \epsilon_2\mu_1 + 2\gamma_1\gamma_2)(\gamma_1^2 + \gamma_x^2) - \epsilon_1\epsilon_2(\mu_1^2 - \mu_x^2) \\
&\quad - \mu_1\mu_2(\epsilon_1^2 - \epsilon_x^2)](\sin\alpha)^2, \\
D_1 &= -4\gamma_1(\epsilon_1\mu_x + \epsilon_x\mu_1)(\epsilon_2\mu_2 - \gamma_2^2)\cos\alpha, \\
D_0 &= (\epsilon_2\mu_2 - \gamma_2^2)\{[(\epsilon_1\mu_1 + \epsilon_x\mu_x) - (\gamma_1^2 + \gamma_x^2)]^2 \\
&\quad - (\epsilon_1\mu_x + \epsilon_x\mu_1)^2\}, \quad (14)
\end{aligned}$$

where α is the angle between the wave normal $\hat{\mathbf{k}}$ and the direction of the constant magnetic field $\mathbf{B}_0$. In general, this quadratic equation can have four different solutions for the index n, but we can show that, for small values of ϵ_x, μ_x, and γ, two values of n are positive and the other two are negative according to Descartes' rule of signs.

Since D_3 and D_1 are odd functions of $\sin\alpha$, we expect the values of n to be nonsymmetrical in opposite directions of propagation (when $\alpha \to \pi + \alpha \Rightarrow D_3 \to -D_3$ and $D_1 \to -D_1$). This nonsymmetrical effect can be used to build or improve nonreciprocal optical devices. It vanishes when either the chirality or the gyrotropy vanishes.

The quartic dispersion equation is rendered quadratic when $D_3 = D_1 = 0$, which is satisfied if one of the following conditions occurs:

(a) $\gamma_1 = \gamma_2 = 0$, which occurs in either a nonchiral medium or in a crystal of class *11*, *16*, or *23* according to Voight's classification,[48]
(b) $\epsilon_x = \mu_x = 0$, which occurs in the absence of the Faraday effect,
(c) $(\epsilon_1/\mu_1) = (\epsilon_2/\mu_2) = -(\epsilon_x/\mu_x)$, which occurs in an initially singly refracting medium of a spherical or ellipsoidal index vector surface that is placed in a constant magnetic field,[49]
(d) $(\epsilon_1/\mu_1) = -(\epsilon_x/\mu_x)$ and $\gamma_1 = 0$.

For comparison, Fig. 1 plots the index vector surface that has rotational symmetry around the $\hat{c}$ axis for a general case and for special cases. Analytical solutions for the dispersion equation can be obtained in the two special cases when the wave propagates either perpendicular or parallel to the constant magnetic field.

When the wave vector is perpendicular to the magnetic field ($\mathbf{k}\perp\hat{\mathbf{c}}$), $\cos\alpha = 0$ and the dispersion equation becomes quadratic ($D_1 = D_3 = 0$), yielding two symmetrical values for n. Equations (14) become

$$\begin{aligned}
D_4 &= (\epsilon_1\mu_1 - \gamma_1^2), \quad D_3 = D_1 = 0, \\
D_2 &= -2\gamma_1\gamma_2(\epsilon_1\mu_1 + \epsilon_x\mu_x) + (\epsilon_1\mu_2 + \epsilon_2\mu_1 + 2\gamma_1\gamma_2) \\
&\quad \times (\gamma_1^2 + \gamma_x^2) - \epsilon_1\epsilon_2(\mu_1^2 - \mu_x^2) \\
&\quad - \mu_1\mu_2(\epsilon_1^2 - \epsilon_x^2), \\
D_0 &= (\epsilon_2\mu_2 - \gamma_2^2)\{[(\epsilon_1\mu_1 + \epsilon_x\mu_x) - (\gamma_1^2 + \gamma_x^2)]^2 \\
&\quad - (\epsilon_1\mu_x + \epsilon_x\mu_1)^2\}. \quad (15)
\end{aligned}$$

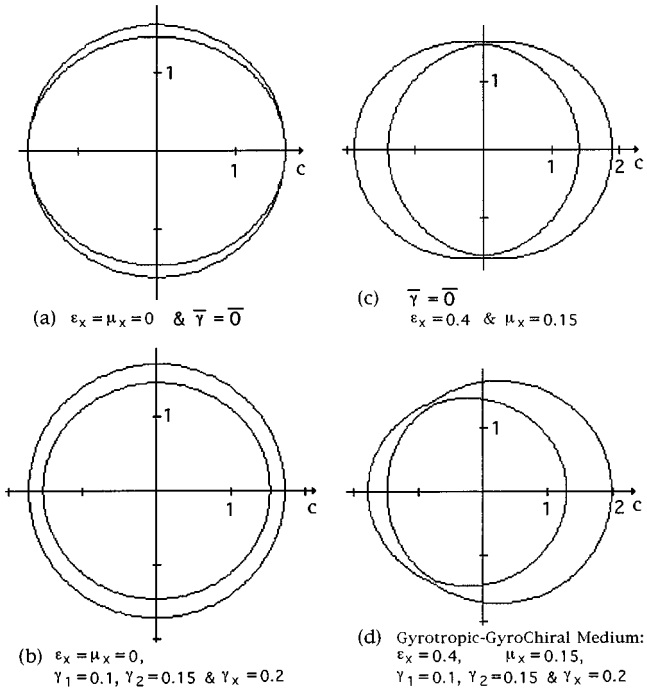

Fig. 1. Refractive-index surfaces for a gyrotropic–gyrochiral medium when $\epsilon_1 = 2.5$, $\epsilon_2 = 2.3$, $\mu_1 = 1.08$, and $\mu_2 = 1.05$ (a) in the absence of optical activity, (b) in the absence of the Faraday effect, (c) in the absence of chirality, and (d) in a general case.

The solution of the dispersion equation is $n^2 = [-D_2 \pm (D_2{}^2 - 4D_4D_0)^{1/2}]/2D_4$, but it involves many terms that do not allow for a simple observation for the effect of each element separately. We can prove, however, that this direction is not an optical axis even if $\bar{\gamma} = \bar{0}$.

5. WAVE PROPAGATION IN THE DIRECTION OF THE MAGNETIC FIELD

When the plane wave propagates in the direction of the applied magnetic field, $\sin\alpha = 0$ and the solutions will be given in terms of $\cos\alpha = \pm 1$ because it covers both cases when $\hat{\mathbf{k}}$ is parallel and antiparallel to $\hat{\mathbf{c}}$.

A. Refractive Indices

Substituting $\sin\alpha = 0$ into Eqs. (14), we get

$$D_4 = (\epsilon_2\mu_2 - \gamma_2{}^2), \quad D_3 = 0,$$

$$D_2 = -2(\epsilon_2\mu_2 - \gamma_2{}^2)[(\gamma_1{}^2 - \gamma_x{}^2) + (\epsilon_1\mu_1 + \epsilon_x\mu_x)],$$

$$D_1 = -4\gamma_1(\epsilon_1\mu_x + \epsilon_x\mu_1)(\epsilon_2\mu_2 - \gamma_2{}^2)\cos\alpha,$$

$$D_0 = (\epsilon_2\mu_2 - \gamma_2{}^2)\{[(\epsilon_1\mu_1 + \epsilon_x\mu_x) - (\gamma_1{}^2 + \gamma_x{}^2)]^2 - (\epsilon_1\mu_x + \epsilon_x\mu_1)^2\}. \quad (16)$$

Substituting these terms into Eq. (13) and dropping the factor, $(\epsilon_2\mu_2 - \gamma_2{}^2)$, we get

$$n^4 - 2[(\gamma_1{}^2 - \gamma_x{}^2) + (\epsilon_1\mu_1 + \epsilon_x\mu_x)]n^2 - 4\gamma_1(\epsilon_1\mu_x + \epsilon_x\mu_1)\cos\alpha\, n + [(\epsilon_1\mu_1 + \epsilon_x\mu_x) - (\gamma_1{}^2 + \gamma_x{}^2)]^2 - (\epsilon_1\mu_x + \epsilon_x\mu_1)^2 = 0, \quad (17)$$

which can be factorized to yield the following two equations:

$$n^2 - 2\gamma_1\cos\alpha\, n - (\epsilon_1\mu_x + \epsilon_x\mu_1) - (\epsilon_1\mu_1 + \epsilon_x\mu_x) + (\gamma_1{}^2 + \gamma_x{}^2) = 0, \quad (18a)$$

$$n^2 + 2\gamma_1\cos\alpha\, n + (\epsilon_1\mu_x + \epsilon_x\mu_1) - (\epsilon_1\mu_1 + \epsilon_x\mu_x) + (\gamma_1{}^2 + \gamma_x{}^2) = 0. \quad (18b)$$

Equations (18) can yield three different solutions:

$$n_1 = \gamma_1\cos\alpha + \sqrt{\Delta_1} \qquad \Delta_1 > 0, \quad \sqrt{\Delta_1} > -\gamma_1\cos\alpha, \quad (19a)$$

$$n_2 = -\gamma_1\cos\alpha + \sqrt{\Delta_2} \qquad \Delta_2 > 0, \quad \sqrt{\Delta_2} > \gamma_1\cos\alpha, \quad (19b)$$

$$n_3 = \begin{cases} \gamma_1\cos\alpha - \sqrt{\Delta_1} & \Delta_1 > 0, \ \gamma_1\cos\alpha > \sqrt{\Delta_1} > 0 \\ -\gamma_1\cos\alpha - \sqrt{\Delta_2} & \Delta_2 > 0, \ \gamma_1\cos\alpha < -\sqrt{\Delta_2} < 0 \end{cases}, \quad (19c)$$

where

$$\Delta_1 = (\epsilon_1 + \epsilon_x)(\mu_1 + \mu_x) - \gamma_x{}^2, \quad (20a)$$

$$\Delta_2 = (\epsilon_1 - \epsilon_x)(\mu_1 - \mu_x) - \gamma_x{}^2. \quad (20b)$$

From Eqs. (19) we observe that any combination of the refractive indices n_1, n_2, and n_3 can exist, depending on the values of ϵ_1, ϵ_x, μ_1, μ_x, γ_1, and γ_x. In particular, we obtain trirefringence, say, for $\cos\alpha = +1$, if the following condition is satisfied:

$$0 < \Delta_1 < \gamma_1{}^2 < \Delta_2, \quad (21)$$

which necessitates that $(\epsilon_1\mu_x + \epsilon_x\mu_1) < 0$ and $(\epsilon_1\mu_1 + \epsilon_x\mu_x) > \gamma_x{}^2$. This is possible when γ_1 has values that are comparable to $\sqrt{\epsilon_1\mu_1}$ and when $\epsilon_x\mu_x < 0$. However, we can easily show that, when trirefringence occurs in a certain direction of propagation, only one plane wave can propagate in the opposite direction. In addition, we should take some risk in mentioning that in this case two of three refractive indices are close to zero. If this is physically possible, it has many scientific implications, and for this reason it deserves more attention.

It is also worth mentioning that, from symmetry considerations around the axis $\hat{\mathbf{c}}$, the values of ϵ_2, μ_2, and γ_2 have no bearing on the existence of the refractive indices, although they affect the shapes of the refractive-index surfaces.

B. Directions of the Field Vectors and Wave Polarizations

To determine the direction of the electric field vector for each of the refractive indices n_1 and n_2 that are given in Eqs. (19), we first introduce the following parameters:

$$a = (\epsilon_1\mu_1 + \epsilon_x\mu_x), \qquad d = (\gamma_1{}^2 + \gamma_x{}^2),$$

$$b = (\epsilon_1\mu_x + \epsilon_x\mu_1),$$

$$d_1d_2 = (\epsilon_1{}^2 - \epsilon_x{}^2)(\mu_1{}^2 - \mu_x{}^2) = a^2 - b^2. \quad (22)$$

In terms of these parameters, Eqs. (18) can be rewritten as follows:

$$(\text{for } n_1) \qquad n^2 - 2\gamma_1\cos\alpha\, n - a - b + d = 0, \quad (23a)$$

$$(\text{for } n_2) \qquad n^2 + 2\gamma_1\cos\alpha\, n - a + b + d = 0. \quad (23b)$$

To show that $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to $\hat{\mathbf{c}}$ (or to $\mathbf{k}$), we dot multiply Eqs. (4) by $\hat{\mathbf{c}}$, set the result to zero [after Eq. 1(c)], and get

$$\epsilon_2(\hat{\mathbf{c}}\cdot\mathbf{E}) + i\gamma_2(\hat{\mathbf{c}}\cdot\mathbf{H}) = -i\gamma_2(\hat{\mathbf{c}}\cdot\mathbf{E}) + \mu_2(\hat{\mathbf{c}}\cdot\mathbf{H}) = 0,$$

which leads to

$$(\epsilon_2\mu_2 - \gamma_2{}^2)(\hat{\mathbf{c}}\cdot\mathbf{E}) = (\epsilon_2\mu_2 - \gamma_2{}^2)(\hat{\mathbf{c}}\cdot\mathbf{H}) = 0. \quad (24)$$

Therefore, for $(\epsilon_2\mu_2 - \gamma_2{}^2) \neq 0$,

$$\hat{\mathbf{c}}\cdot\mathbf{E} = \hat{\mathbf{c}}\cdot\mathbf{H} = 0. \quad (25)$$

Hence $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to the direction of propagation. Now, let

$$\bar{\mathbf{A}} = (\bar{\mathbf{I}} + \bar{\mathbf{M}}\cdot\bar{\mathbf{N}}), \quad (26)$$

where $\bar{\mathbf{N}}$ and $\bar{\mathbf{M}}$ are given by Eqs. (9). Then the homogeneous equation [Eq. (8a)] becomes

$$\bar{\mathbf{A}}\cdot\mathbf{E} = \mathbf{0}, \quad (27)$$

and it provides the solution for the electric field vector. Because $|\bar{\mathbf{A}}| = 0$ (that is, the dispersion equation), we can write[26]

$$\bar{\mathbf{A}}\cdot\text{adj}\,\bar{\mathbf{A}} = |\bar{\mathbf{A}}|\bar{\mathbf{I}} = \bar{\mathbf{0}}. \quad (28)$$

Two possible cases can exist, as follows.

1. *adj($\bar{A}$)$\neq\bar{0}$ ($\bar{A}$ is planar)*

Postmultiplying Eq. (28) by an arbitrary vector $\mathbf{u}$, we get

$$\bar{\mathbf{A}} \cdot [(\text{adj } \bar{\mathbf{A}}) \cdot \mathbf{u}] = \mathbf{0}. \tag{29}$$

Comparing Eq. (29) with Eq. (27), we obtain the direction of $\mathbf{E}$:

$$\mathbf{e} = (\text{adj } \bar{\mathbf{A}}) \cdot \mathbf{u}. \tag{30}$$

As $\mathbf{u}$ is an arbitrary vector that does not make $(\text{adj } \bar{\mathbf{A}}) \cdot \mathbf{u} = \mathbf{0}$, we choose $\mathbf{u} \perp \hat{\mathbf{c}}$, evaluate $(\text{adj } \bar{\mathbf{A}}) \cdot \mathbf{u}$, and arrange real and imaginary factors. We get

$$\mathbf{e} = X\mathbf{u} - iY(\hat{\mathbf{c}} \times \mathbf{u}), \tag{31}$$

where

$$X = d_1 d_2 - a(n^2 + d) - 2b\gamma_1 \cos \alpha\, n, \tag{32a}$$

$$Y = b(n^2 + d) + 2b\gamma_1 \cos \alpha\, n. \tag{32b}$$

Substituting Eqs. (23) into Eqs. (32), we get

$$(\text{for } n_1) \qquad Y = -X = (a + b)(b + 2\gamma_1 \cos \alpha\, n), \tag{33a}$$

$$(\text{for } n_2) \qquad Y = -X = (a - b)(b + 2\gamma_1 \cos \alpha\, n). \tag{33b}$$

Therefore, after substituting Eqs. (33) into Eq. (31) and dropping the nonzero scalar factor, we find the directions of the electric field vectors:

$$(\text{for } n_1) \qquad \mathbf{e}_1 = \mathbf{u} + i(\hat{\mathbf{c}} \times \mathbf{u}), \tag{34a}$$

$$(\text{for } n_2) \qquad \mathbf{e}_2 = \mathbf{u} - i(\hat{\mathbf{c}} \times \mathbf{u}). \tag{34b}$$

The directions of the magnetic field vectors can also be found by use of the following identity that is derived from ME's and the CR's:

$$\mathbf{H} = (c\mu_0\bar{\mu})^{-1}[i\bar{\bar{\gamma}} + n(\mathbf{k} \times \bar{\mathbf{I}})]\} \cdot \mathbf{E} = \mathbf{0}. \tag{35}$$

They are given by

$$(\text{for } n_1) \qquad \mathbf{h}_1 = B_1[\mathbf{u} + i(\hat{\mathbf{c}} \times \mathbf{u})], \tag{36a}$$

$$(\text{for } n_2) \qquad \mathbf{h}_2 = B_2[\mathbf{u} - i(\hat{\mathbf{c}} \times \mathbf{u})], \tag{36b}$$

where

$$B_{1,2} = i[c\mu_0(\mu_1 \pm \mu_x)]^{-1}[\gamma_1 + n \cos \alpha \pm i\gamma_x]. \tag{37}$$

Therefore the two electric fields are circularly polarized and rotate in opposite directions. In addition, because $\mathbf{e}_1 \cdot \mathbf{e}_2^* = 0$, the two electric field vectors are orthogonal to each other, and they are also perpendicular to the direction of propagation. For $\cos \alpha = +1$, $\mathbf{e}_1$ and $\mathbf{e}_2$ are right-handed and left-handed circularly polarized, respectively. For $\cos \alpha = -1$, $\mathbf{e}_1$ and $\mathbf{e}_2$ are left-handed and right-handed circularly polarized, respectively.

2. adj $\bar{A}$=$\bar{0}$ ($\bar{A}$ is linear)

$\bar{\mathbf{A}}$ must be represented by a single dyad, that is, $\bar{\mathbf{A}} = \mathbf{ab}$. After expanding $\bar{\mathbf{A}}$, we find that it can be written in the form $\bar{\mathbf{A}} = A\hat{\mathbf{c}}\hat{\mathbf{c}} + B\bar{\mathbf{I}} + C(\hat{\mathbf{c}} \times \bar{\mathbf{I}})$. Conditions for the linearity of $\bar{\mathbf{A}}$ lead us to the following expressions:

$$(\text{for } n_1) \qquad (a + b)(b + 2\gamma_1 \cos \alpha\, n) = 0, \tag{38a}$$

$$(\text{for } n_2) \qquad (a - b)(b + 2\gamma_1 \cos \alpha\, n) = 0, \tag{38b}$$

which are the scalar factors that we considered nonzero in Eqs. (33). We observe the following:

(a) If $(b + 2\gamma_1 \cos \alpha\, n) = 0$, then, according to Eqs. (23), the two refractive indices n_1 and n_2 are equal. Therefore $\hat{\mathbf{c}}$ becomes an optical axis in one direction but $(-\hat{\mathbf{c}})$ does not. In this case the wave may have any polarization and is subject only to ME's.

(b) If $a + b = (\epsilon_1 + \epsilon_x)(\mu_1 + \mu_x) = 0$ (for n_1) then, according to Eqs. (19) and (20), n_1 does not exist. In this case, n_1 is in the cutoff mode.

(c) If $a - b = (\epsilon_1 - \epsilon_x)(\mu_1 - \mu_x) = 0$ (for n_2), then, according to Eqs. (19) and (20), n_2 does not exist. In this case, n_2 is in the cutoff mode.

C. Poynting's Vector and Energy Density

Letting the electric field vector be given by $\mathbf{E} = A\mathbf{e}$; substituting Eqs. (34) and (36) into the time-averaged Poynting (or power density) vector identity

$$\langle \mathbf{P} \rangle = 1/2\ \text{Re}(\mathbf{E} \times \mathbf{H}^*), \tag{39}$$

we obtain

$$\begin{aligned} \langle \mathbf{P}_{1,2} \rangle &= \text{Re}(-i|A|^2 B^*)\hat{\mathbf{c}} \\ &= |A|^2[c\mu_0(\mu_1 \pm \mu_x)]^{-1} \\ &\quad \times (n_{1,2} \cos \alpha \mp \gamma_1)\hat{\mathbf{c}}. \end{aligned} \tag{40}$$

Substituting for the values of $n_{1,2}$ from Eqs. (19) into Eq. (40), we get

$$\langle \mathbf{P}_{1,2} \rangle = |A|^2[c\mu_0(\mu_1 \pm \mu_x)]^{-1}[\cos \alpha\sqrt{\Delta_{1,2}}]\hat{\mathbf{c}}. \tag{41}$$

Therefore, when $\mathbf{k}$ is parallel to $\hat{\mathbf{c}}$, the power is transmitted in the same direction. In addition, the power that is carried by either wave is the same in opposite directions in spite of the nonsymmetry of the refractive indices. This fact is useful when phase differences are sought without compromising the symmetry of the power density.

We can show that, in an optically active medium that is characterized by the CR's in Eqs. (4), the average energy density is given by[50]

$$\begin{aligned} \langle W \rangle &= \frac{1}{2}\frac{\partial}{\partial \omega}[\omega\ \text{Re}(\mathbf{E} \cdot \mathbf{D}^*)] \\ &= \frac{1}{2}\left(\epsilon_0 \mathbf{E}^* \cdot \frac{\partial}{\partial \omega}[\omega\bar{\bar{\epsilon}}(\omega)] \cdot \mathbf{E} \right. \\ &\quad \left. - \frac{1}{c}\,\text{Im}\left\{\mathbf{E}^* \cdot \frac{\partial}{\partial \omega}[\omega\bar{\bar{\gamma}}(\omega)] \cdot \mathbf{H}\right\}\right). \end{aligned} \tag{42}$$

Reevaluating Eq. (42) for $\mathbf{k} \| \hat{c}$ and making use of Eq. (37), we get

$$\begin{aligned} \langle W_{1,2} \rangle = \epsilon_0|A|^2 \Bigg\{ &\frac{\partial[\omega(\epsilon_1 \pm \epsilon_x)]}{\partial \omega} + (\mu_1 \pm \mu_x)^{-1} \\ &\times \left[\frac{\partial(\omega\gamma_1)}{\partial \omega}(\gamma_1 \pm n_{1,2} \cos \alpha) - \frac{\partial(\omega\gamma_x)}{\partial \omega}\gamma_x\right]\Bigg\}. \end{aligned} \tag{43}$$

Knowledge of the dependency of the elements on the frequency is expected before the average energy density can be evaluated.

The energy velocity, which is given by $\mathbf{V}_E = \langle \mathbf{P} \rangle / \langle W \rangle$, can easily be evaluated after the energy density is known.

Its direction is along the real-time average of the Poynting vector, which is parallel to the direction of propagation in this case.

6. SUMMARY

In this paper we have investigated the characteristics of electromagnetic wave propagations in linear, homogeneous, source-free, lossless, biuniaxial, gyrochiral, and gyrotropic media. The dispersion equation was derived, the nonsymmetrical values of the refractive indices in opposite directions of propagation were observed, and the effects of the various parameters were discussed. In addition, closed-form solutions for the refractive indices and the field vectors were derived for waves propagating in the direction of the applied constant magnetic field. We also showed that waves propagating in this direction are circularly polarized, except when the two refractive indices are equal. In such a case there is no restriction on the wave polarization. Furthermore, we showed that, theoretically, it is possible for trirefringence to occur. If trirefringence is physically achieved, it will have far-reaching effects and many applications. In addition, the energy equations were derived.

REFERENCES

1. N. Engheta and D. L. Jaggard, "Electromagnetic chirality and its applications," IEEE Antennas Propag. Soc. Newsletter **30**, 6–12 (1988).
2. A. Lakhtakia, "Recent contributions to classical electromagnetic theory of chiral media: what next?," Speculations Sci. Technol. **14**, 2–17 (1991).
3. F. Mariotte and N. Engheta, "Reflection and transmission of guided electromagnetic waves at an air–chiral interface and at a chiral slab in a parallel-slate wave-guide," IEEE Trans. Microwave Theory Tech. **18**, 1895–1906 (1993).
4. A. K. Singh, P. Khastgir, and O. N. Singh, "Modal characteristics of EM wave-propagation through achirofibre with different core and cladding chiral admittances," Opt. Commun. **115**, 256–260 (1995).
5. A. R. Samant and K. W. Whites, "Electromagnetic-wave propagation in a chiral-material-filled rectangular waveguide," Microwave Opt. Technol. Lett. **8**, 106–111 (1995).
6. R. C. Qiu and I. T. Lu, "Guided-waves in chiral optical fibers," J. Opt. Soc. Am. A **11**, 3212–3219 (1994).
7. W. Y. Yin, W. B. Wang, and P. Li, "Guided electromagnetic-waves in gyrotropic chirowaveguides," IEEE Trans. Microwave Theory Tech. **42**, 2156–2163 (1994).
8. A. K. Singh, K. S. Singh, P. Khastgir, S. P. Ojha, and O. N. Singh, "Modal cutoff condition of an optical chiral fiber with different chiralities in the core and the cladding," J. Opt. Soc. Am. B **11**, 1283–1287 (1994).
9. A. J. Viitanen and I. V. Lindell, "Uniaxial chiral quarter-wave polarization transformer," Electron. Lett. **29**, 1074–1075 (1993).
10. K. M. Johnson, D. J. McKnight, and I. Underwood, "Smart spatial light modulators using liquid-crystals on silicon," IEEE J. Quantum Electron. **29**, 699–714 (1993).
11. K. E. Susse, S. Wozniak, and G. Wagniere, "Chiral media for ultrafast modulation of light by light," Opt. Commun. **100**, 374–380 (1993).
12. P. A. Schultz, "Wavelength independent Faraday isolator," Appl. Opt. **28**, 4458–4464 (1989).
13. W. T. Astbury, "Optical rotatory power of solid crystals," in *International Critical Tables*, E. Washburn, ed. (McGraw-Hill, New York, 1930), Vol. 7, pp. 353–354.
14. Y. W. Yin, P. Li, and W. Wan, "Spectral-domain decoupling method of electromagnetic waves propagating and radiating in bianisotropic gyroelectromagnetic chiral media," Microwave Opt. Technol. Lett. **7**, 187–193 (1994).
15. P. Koivisto, "Polarization properties of plane-waves in transversely bianisotropic uniaxial medium," Microwave Opt. Technol. Lett. **6**, 858–862 (1993).
16. A. J. Viitanen and I. V. Lindell, "Plane-wave propagation in a uniaxial bianisotropic medium with an application to a polarization transformer," Int. J. Infrared Millimeter Waves **14**, 1993–2010 (1993).
17. I. V. Lindell, S. A. Tretyakov, and A. J. Viitanen, "Plane-wave propagation in a uniaxial chiro-omega medium," Microwave Opt. Technol. Lett. **6**, 517–520 (1993).
18. S. A. Tretyakov and A. A. Sochava, "Reflection and transmission of plane electromagnetic waves in uniaxial bianisotropic materials," Int. J. Infrared Millimeter Waves **15**, 829–856 (1994).
19. P. Hillion, "Electromagnetism in anisotropic chiral media," Phys. Rev. **47**, 1365–1374 (1993).
20. H. Taouk, "Optical wave propagation in active media," Ph.D dissertation (Ohio University, Athens, Ohio, 1991).
21. N. Engheta and D. L. Jaggard, "Electromagnetic waves in Faraday chiral media," IEEE Trans. Antennas Propag. **40**, 967–973 (1992).
22. M. P. Silverman and R. B. Sohn, "Effects of circular birefringence on light propagation and reflection," Am. J. Phys. **54**, 69–76 (1986).
23. W. Y. Yin and P. Li, "The field distribution and dispersion characteristics in chiroferrite central-loaded wave-guide," Int. J. Infrared Millimeter Waves **15**, 451–463 (1994).
24. C. M. Krowne, "Electromagnetic properties of nonreciprocal composite chiral-ferrite media," IEEE Trans. Antennas Propag. **41**, 1289–1295 (1993).
25. Z. X. Shen, "The theory of chiroferrite wave-guides," Microwave Opt. Technol. Lett. **6**, 397–401 (1993).
26. H. Chen, *Theory of Electromagnetic Waves. A Coordinate Free Approach* (McGraw-Hill, New York, 1983), Chaps. 2 and 3, pp. 58–122.
27. F. I. Federov, "On the theory of optical activity in crystals. I. The law of conservation of energy and the optical activity tensors," Opt. Spectrosc. (USSR) **6**, 49–53 (1959).
28. M. Born, *Optik* (Springer-Verlag, Berlin, 1933), Chap. 7, p. 298.
29. L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, New York, 1960), Chap. 11, p. 313.
30. E. J. Post, *Formal Structure of Electromagnetics* (Wiley, New York, 1962), Chap. 8, p. 160.
31. R. M. Peterson, "Comparison of two theories of optical activity," Am. J. Phys. **43**, 969–972 (1975).
32. D. Eimerl, "Quantum electrodynamics of optical activity in birefringent crystals," J. Opt. Soc. Am. B **5**, 1453–1461 (1988).
33. M. P. Silverman, "Reflection and refraction at the surface of a chiral medium: comparison of gyrotropic constitutive relations invariant under duality transformation," J. Opt. Soc. Am. A **3**, 830–837 (1986).
34. C. F. Bohren, "Scattering of electromagnetic waves by an optically active spherical shell," J. Chem. Phys. **62**, 1566–1571 (1975).
35. R. A. Satten, "Time reversal symmetry and electromagnetic polarization fields," J. Chem. Phys. **28**, 742–743 (1958).
36. E. G. Cullwick, *Electromagnetism and Relativity* (Longnams, Green, New York, 1957), Chap. 2, p. 18.
37. V. Rojanski, *Electromagnetic Fields and Waves* (Prentice-Hall, Englewood Cliffs, N.J., 1971), Chap. 20, p. 103.
38. R. W. P. King, "Mathematical description of space: formation of field equations," in *Electromagnetic Engineering*, 1st ed., Radio Communication Series. I. Fundamentals (McGraw-Hill, New York, 1945), Vol. 1, pp. 108–113.
39. G. Bekefi and A. H. Brett, *Electromagnetic Vibrations, Waves, and Radiation* (MIT Press, Cambridge, Mass., 1977), Chap. 6, p. 395.
40. P. Visscher, *Fields and Electrodynamics* (Wiley, New York, 1988), Chap. 23, p. 470.

41. J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), Chap. 6, Sec. 11, p. 245.
42. D. Polder, "On the theory of ferromagnetic resonance," Philos. Mag. **40**, 99–115 (1949).
43. S. Ramaseshan and V. Sivaramakrishnan, "Faraday effect in diamagnetic crystals," in *Progress in Crystal Physics*, R. S. Krishnan, ed. (Interscience, New York, 1958), Vol. 1, pp. 168–183.
44. J. F. Nye, *Physical Properties of Crystals* (Clarendon, Oxford, 1969), Chap. 14, p. 260.
45. M. S. Kluskens and E. H. Newman, "Microstrip lines on a chiral substrate," IEEE Trans. Microwave Theory Tech. **39**, 1889–1891 (1991).
46. I. V. Lindell, A. H. Sihvola, P. Puska, and L. H. Ruotanen, "Conditions for the parameter dyadics of lossless bianisotropic media," Microwave Opt. Technol. Lett. **8**, 268–272 (1995).
47. W. J. Tabor and F. S. Chen, "Electromagnetic propagation through materials possessing both Faraday rotation and birefringence: experiments with ytterbium orthoferrite," J. Appl. Phys. **40**, 2760–2765 (1969).
48. W. Voight, *Lehrbuch der Kristallphysik*, 1st ed. (Teubner, Leipzig, 1910), Chap. 1, p. 15.
49. H. Taouk, "Wave propagation in general anisotropic media," M.S. thesis (Ohio University, Athens, Ohio, 1986).
50. A. Tonning, "Energy density in continuous electromagnetic media," IRE Trans. Antennas Propag. **AP-8**, 428–434 (1960).