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Bifurcation, stability, and critical slowing down in a simple mass–spring system

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Dedicated to Professor Brian Straughan, in celebration of his 75th birthday.

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ABSTRACT

We analyse a simple mass–spring system as an accessible context for showcasing how continuous changes to system parameters can lead to critical transitions ('tipping points'). Two kinds of transition are explored in particular: saddle–node bifurcations, due to changes in a mass forcing parameter a; and pitchfork bifurcations, due to changes in a spring separation parameter X. Both types of bifurcation arise as features of a cusp catastrophe characterised in $X - a$ parameter space by the critical curve $X^{2/3} + a^{2/3} = 1$, leading to hysteresis cycles, as described by C. Ong (2021), and non-reversible pitchfork catastrophes, which are discussed here for the first time. In each case we demonstrate critical slowing down of the oscillation period $\tau \to \infty$ as the system approaches bifurcation.

1. Introduction

Bifurcation theory may be used to characterise a wide range of phenomena in dynamical systems [1], and has applications to a broad spectrum of scientific disciplines, including climate modelling [2], ecosystem science [3], structural mechanics [4–6], financial economics [7], social dynamics [8,9], and anatomy [10]. Indeed, since bifurcation theory provides a way of accounting for how continuous changes to system variables can lead to abrupt changes in system dynamics, it is often used to model critical transitions or 'tipping points' [8,11]. Despite these varied applications, however, the literature describing simple practical contexts for experimenting on system bifurcation remains relatively sparse [12,13].

Recently, Ong has helped to address this gap in the literature by presenting an analysis of a simple mass–spring system suitable for practical demonstrations of critical transitions [13]. In particular, he has shown that variations in an external forcing parameter can lead to saddle–node bifurcations and hysteresis. Ong's treatment is well motivated and compelling, and represents a valuable addition to the literature on how critical transitions can be explored in a laboratory setting using basic equipment [13]. Nevertheless, by focusing on hystereses, Ong overlooks a range of other phenomena—such as pitchfork bifurcations and critical slowing down—that are also exhibited by his system, restricting its illustrative power.

In this article, therefore, we substantially extend Ong's analysis with a view to exploiting the full potential of the system he proposes, and in so doing show how the hysteresis he describes is simply one aspect of a much wider set of features associated with a cusp catastrophe [1]. To this end the paper is structured as follows. Beginning with a brief review of the system (Section 2), we develop Ong's procedure for determining the critical bifurcation values by presenting new algebraic expressions for the equilibrium states. Note that whilst Ong's approach assumes four control variables, here we introduce a more general dimensionless framework to show that bifurcation is in fact determined by two parameters only—a spring separation parameter X, and a spring forcing parameter a—such that the number of equilibria are characterised by a critical curve $X^{2/3} + a^{2/3} = 1$ (Section 3). In agreement with Ong, we find that variation in a results in saddle–node bifurcations and hysteresis (Section 4) [13]. Nevertheless, whilst Ong's analysis is (in effect) limited to variations in a, here we go deeper by showing how changes to X can also lead to critical transitions in the form of pitchfork bifurcations (Section 5). In either case (saddle–node or pitchfork), we complete our analysis by deriving a linear theory for the stability of the system at bifurcation (Section 6), including a closed form expression for the system's oscillation period τ [14]. Crucially, this procedure allows us to demonstrate blow-up of the oscillation period ($\tau \to \infty$) as the system approaches bifurcation, a phenomenon known as critical slowing down [15].

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2. Basic model and system equilibria

The system proposed by Ong is depicted in Fig. 1: two identical springs, each of natural length l_0 and spring constant k, are attached to points separated by a horizontal distance $2x$, and connected in a symmetrical V-shaped configuration with a mass m. The mass is constrained to move in the y-direction only, where y is the vertical displacement, and subject to an on-axis force f, which accounts for any applied forces, including the weight [13]. The total force $F(y)$ on the mass is therefore assumed to act in the y-direction only, and is given by

$$F(y; x, f, k, l_0) = f - 2ky\left[1 - \frac{l_0}{\sqrt{x^2 + y^2}}\right]. \tag{1}$$

Ong studies this equation in terms of four parameters x, f, k, and l_0 [13]. Here we adopt the dimensionless notation

$$A \equiv \frac{F}{2kl_0}, \quad a \equiv \frac{f}{2kl_0}, \quad X \equiv \frac{x}{l_0}, \quad \text{and} \quad Y \equiv \frac{y}{l_0}, \tag{2}$$

where A is the normalised total force, a is an applied forcing parameter, X is a spring separation parameter, and Y is the normalised displacement. In this way we reduce the number of control parameters from four (x, f, k, and l_0) to two (X and a), such that Eq. (1) is simplified as

$$A(Y; X, a) = a - Y\left[1 - (X^2 + Y^2)^{-1/2}\right]. \tag{3}$$

The system is in equilibrium when the total force on the mass is $A(Y) = 0$; thus, the equilibrium positions $Y = Y_*$ correspond to the roots of $A(Y)$ (see Figs. 2 and 3). It follows that a given equilibrium Y_* is stable if $A(Y)$ is decreasing through Y_*, and unstable if $A(Y)$ is increasing through Y_*, where the derivative of $A(Y)$ is

$$A'(Y; X) = -\left[1 - X^2(X^2 + Y^2)^{-3/2}\right]. \tag{4}$$

We present a more formal stability analysis in Section 6.

If $X \geq 1$, then the springs are never compressed. In such situations $A'(Y) \leq 0$, and $A(Y)$ has only one root, i.e., a single, stable equilibrium (see Fig. 2). Otherwise, if $X < 1$, then $A(Y)$ has turning points $A'(Y) = 0$ when

$$Y = \pm Y_c, \qquad \text{where} \qquad Y_c = [X^{4/3} - X^2]^{1/2}, \tag{5}$$

with Y_c a local maximum and $-Y_c$ a local minimum (see Fig. 3). As indicated in Fig. 3, the presence of turning points in the curve when $X < 1$ means that in these cases the number of equilibria (roots) depends on the value of a. The graphical process for considering how the number of equilibria changes is described by Ong [13], and may be understood in our dimensionless notation as follows.

The effect of changing a is to translate the $A(Y)$ curve vertically, as depicted in Fig. 3. Thus, when a is very large $A(Y; a)$ has a single positive root, corresponding to a stable equilibrium Y_*. As a is reduced, however, it eventually reaches a critical value (see Appendix)

$$a = a_c(X) = [1 - X^{2/3}]^{3/2}, \tag{6}$$

for which the curve $A(Y; a_c)$ touches the Y-axis at its local minimum $-Y_c$. For this value of a the curve $A(Y; a_c)$ has two roots: a double,

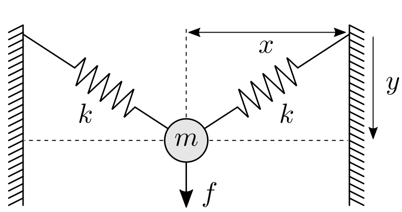

Fig. 1. Ong's V-shaped mass–spring system [13].

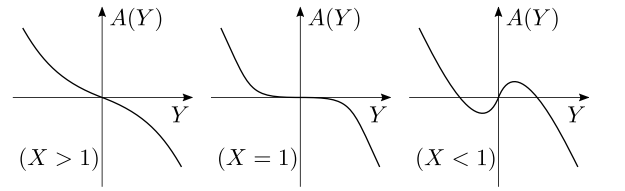

Fig. 2. Force $A(Y)$ as a function of Y for situations with no external forcing, i.e., $a = 0$. $A(Y)$ has a single real root at $Y = 0$ when $X > 1$ (left); a triple root at $Y = 0$ when $X = 1$ (centre); and three distinct roots, $Y = 0$ and $Y = \pm\sqrt{1 - X^2}$, when $X < 1$ (right), cf. Fig. 3.

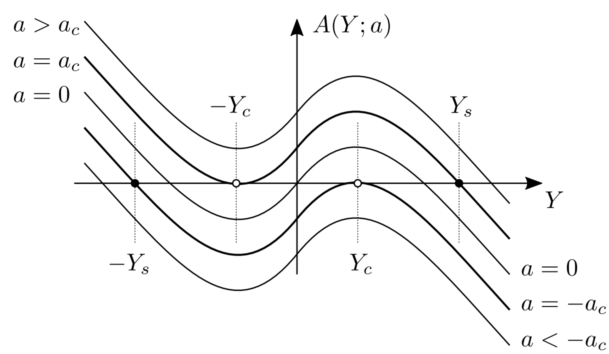

Fig. 3. Force $A(Y)$ as a function of Y for a fixed value of $X < 1$, and five illustrative values of a as labelled. $A(Y)$ has a maximum at $Y = Y_c$ and a minimum at $Y = -Y_c$.

negative root, corresponding to a critical unstable equilibrium $Y_* = -Y_c$; and a distinct, positive root, corresponding to a critical stable equilibrium $Y_* = Y_s$. This second critical equilibrium was not computed by Ong [13], but is given by (see Appendix)

$$Y_s = a_c^{1/3}[1 + (1 - a_c^{2/3} + a_c^{4/3})^{1/2}]. \tag{7}$$

When a is reduced to within the range $a_c > a > -a_c$, the curve $A(Y; a)$ has three roots, corresponding to three equilibria of alternating stability (stable–unstable–stable). Indeed, $a = -a_c$ represents a second critical value for which the curve $A(Y; -a_c)$ touches the Y-axis at the maximum Y_c; for this value of a the system has a distinct negative root, corresponding to a critical stable equilibrium $Y_* = -Y_s$, and a positive, double root corresponding to a critical unstable equilibrium $Y_* = Y_c$. Further reductions in $a < -a_c$ result in a $A(Y; a)$ having a single negative root only (a single stable equilibrium).

3. Critical curve and cusp catastrophe

Ong's analysis focuses on situations for which the spring separation $X < 1$ is fixed, and only the applied force a modified [13]. However, since X and a can both vary, the condition for three equilibria to exist, i.e., $|a| < a_c = [1 - X^{2/3}]^{3/2}$ (see Section 2), can be expressed more generally by observing that the system has three equilibria whenever the parameter values (X, a) satisfy

$$X^{2/3} + a^{2/3} < 1. \tag{8}$$

Similarly, the system has only one equilibrium configuration when $X^{2/3} + a^{2/3} > 1$. As depicted in Fig. 4, therefore, $X - a$ parameter space is divided into two qualitatively different regions by the critical curve

$$X^{2/3} + a^{2/3} = 1; \tag{9}$$

on this curve the system has two equilibria.

In this way the number of equilibria changes as the parameter space coordinates of the system (X, a) cross the critical curve. If the spring separation is held fixed with $X < 1$, but the forcing a varied, then the qualitative change in the system at $a = \pm a_c$ occurs as a saddle–node

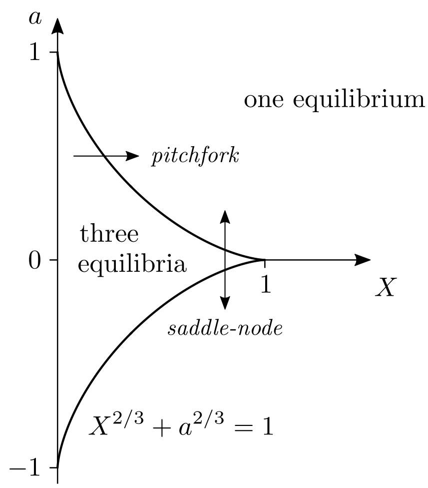

Fig. 4. Critical curve $X^{2/3}+a^{2/3}=1$ dividing $X-a$ parameter space into regions where the system has either one stable equilibrium, or three equilibria of alternating stability; on the curve itself the system has two equilibria of opposite stability. Critical transitions (cusp-catastrophes) occur when traversing the critical curve, either: (i) at saddle–node bifurcations with fixed X (vertical arrows, see Section 4); or (ii) at pitchfork bifurcations with fixed a (horizontal arrow, see Section 5). We note as a point of interest that the critical curve is the *astroid* of the unit circle.

bifurcation (see Section 4). Alternatively, if the forcing is held fixed with $|a|<1$, but the spring separation X varied, then the qualitative change in the system at

$$X = X_c, \qquad \text{where} \qquad X_c(a) = [1-a^{2/3}]^{3/2} \tag{10}$$

occurs as a pitchfork bifurcation (see Section 5). Ong's analysis is limited to a discussion of the saddle–node bifurcations [13]; part of the task of this article, therefore, is to present a treatment of the pitchfork bifurcations.

Observe that our governing Eq. (3) does not include damping; this means that for an arbitrary set of initial conditions, the system will oscillate about stable equilibria indefinitely. In practice, however, all systems are damped, and for this reason our remaining discussion shall assume that any oscillations decay, and that the system comes to rest at a stable equilibrium. As we shall see, one consequence of this is that both forms of bifurcation associated with crossing the critical curve can lead to discontinuous—or 'catastrophic'—transitions in the system's equilibrium configuration. Because the critical curve has a cusp point at $(X,a)=(1,0)$, these transitions are referred to collectively as cusp catastrophes [1].

4. Saddle node bifurcation

To study the saddle–node bifurcations we suppose that the system is in a stable equilibrium Y_* for some fixed $X<1$, and then consider how the equilibrium changes as a is varied. Although Ong does not offer an expression relating Y_* and a explicitly [13], because $A(Y_*)=0$, such a relationship follows immediately from Eq. (3) if Y_* is treated as the independent variable, i.e.,

$$a(Y_*;X) = Y_*\left[1-(X^2+Y_*^2)^{-1/2}\right]. \tag{11}$$

The dependence of Y_* on a may then be visualised by sketching $a(Y_*;X)$ as in Fig. 5, with Y_* on the ordinate axis. This curve describes conditions with three equilibrium configurations of alternating stability when $|a|<a_c$, and conditions with a single equilibrium when $|a|>a_c$.

Let us now consider what happens to the equilibrium configuration Y_* if a is increased from $a<-a_c$ to $a>a_c$. When $a<-a_c$ there is only one equilibrium position, such that the system must occupy a configuration with $Y_*<-Y_c$ negative (see Fig. 5). Provided a is increased sufficiently slowly (so that perturbations are small), the system

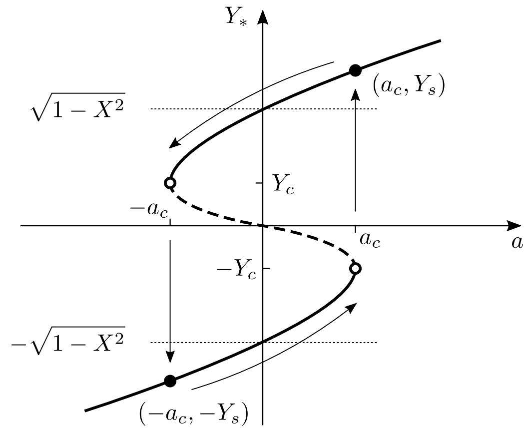

Fig. 5. Stable (solid curves) and unstable (dashed curve) equilibria Y_* as a function of a according to Eq. (11), with $X<1$ fixed. Saddle node bifurcations occur at the critical points $a=\pm a_c$. The system can be made to follow a hysteresis cycle (arrows) by varying a (cf. equilibrium bifurcation in a transversely isotropic elastic solid [6]).

remains on this negative stable curve with Y_* increasing continuously until reaching the saddle–node bifurcation point at $a=a_c$ for which $Y_*=-Y_c$. At this point the negative stable curve terminates, and further increases to $a>a_c$ force the system to undergo a discontinuous transition to a new stable equilibrium configuration with $Y_*\geq Y_s$ positive (see Fig. 5).

Notice that although the system shifts discontinuously from $-Y_c$ to Y_s as a is increased through a_c, the converse is not true: decreasing a through a_c will not return the system to the $Y_*\leq -Y_c$ curve. Indeed, to restore the system to its original state one must reverse the process above in its entirety, i.e., a must be reduced through the saddle–node bifurcation at $a=-a_c$. In this way the system can be made to follow a hysteresis cycle, with discontinuous transitions between positive and negative equilibria Y_* at $a=\pm a_c$ (see Fig. 5), i.e., whenever the system traverses the cusp in $X-a$ space (see Fig. 4).

Observe that the magnitude ΔY_* of these discontinuous transitions—or cusp catastrophes—is

$$\Delta Y_* = Y_s + Y_c = a_c^{1/3}[2-a_c^{2/3}+(1-a_c^{2/3}+a_c^{4/3})^{1/2}], \tag{12}$$

where we used $Y_c = a_c^{1/3}[1-a_c^{2/3}]$ (see Appendix). It may be shown that $0<\Delta Y_*<2$, with transitions of greatest magnitude ($\Delta Y_*\to 2$) corresponding to $a_c\to 1$, i.e., when the spring-separation is minimised ($X\to 0$).

5. Pitchfork bifurcation

To study the pitchfork bifurcations we suppose that the system is in a stable equilibrium Y_* for some fixed $|a|<1$, and then consider how the equilibrium configuration changes as X is varied. We treat two cases in particular: *perfect* pitchfork bifurcations with $a=0$ (see Section 5.1, and Fig. 6); and *imperfect* pitchfork bifurcations with $a\neq 0$ (see Section 5.2, and Fig. 7). Here 'perfect' refers to the odd symmetry of $A(Y;a,X)$ with Y when $a=0$; 'imperfect' refers to the loss of such symmetry when $a\neq 0$. For this reason a is called an imperfection parameter.

5.1. Perfect pitchfork bifurcation ($a=0$)

Ong alludes to the perfect pitchfork bifurcation by observing that if $a=0$, then the number of equilibria (roots of $A(Y)$) changes when $X=1$ (see Fig. 2) [13]. Here we develop this idea by determining the equilibrium configurations. Indeed, solving $A(Y_*;X,0)=0$ for Y_* we have

$$Y_*=0, \qquad \text{and} \qquad Y_* = \pm\sqrt{1-X^2}, \tag{13}$$

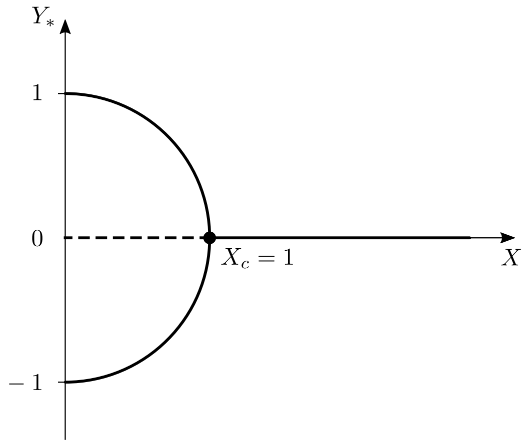

Fig. 6. Stable (solid curves) and unstable (dashed curve) equilibria Y_* as a function of X when $a = 0$. The perfect pitchfork bifurcation occurs at the critical value $X = X_c = 1$. Observe that the springs are at their natural length everywhere on the unit circle $X^2 + Y_*^2 = 1$; they are stretched if $X^2 + Y_*^2 > 1$, and compressed when $X^2 + Y_*^2 < 1$.

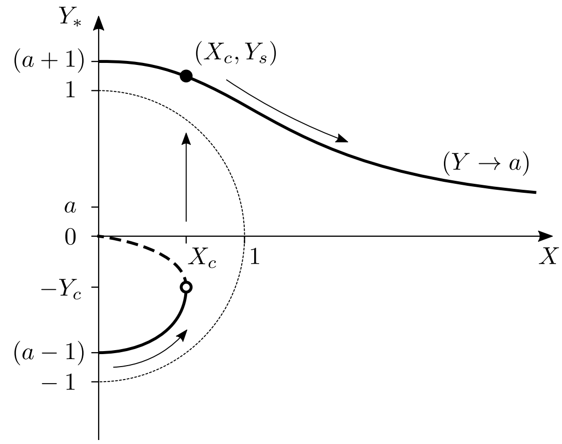

Fig. 7. Stable (solid curves) and unstable (dashed curve) equilibrium solutions for Y_* as a function of X for a fixed positive value of $a < 1$. The imperfect pitchfork bifurcation occurs when $X = X_c = [1 - a^{2/3}]^{3/2}$. As X is increased through X_c, a system in the $Y_* < 0$ stable state will undergo a critical transition (arrows) from $-Y_c < 0$ to $Y_s > 0$. By crossing the unit circle $X^2 + Y_*^2 = 1$ (dotted curve), this transition takes the springs from a compressed state, to an extended state.

such that the number of possible equilibrium configurations increases from one to three as X is decreased through $X_c = 1$ (see Fig. 6).

The physical meaning of these solutions may be understood by observing that the springs are at their natural length $l_0 = \sqrt{x^2 + y^2}$ when $X^2 + Y_*^2 = 1$. Thus, if $X > 1$, then the springs are extended ($X^2 + Y_*^2 > 1$), and—given the absence of applied forcing ($a = 0$)—their elastic tension pulls the mass to a single equilibrium position at the centre of the system $Y_* = 0$. Conversely, if $X < 1$, then symmetry means that two (stable) $Y_* \neq 0$ configurations are possible with the springs relaxed ($X^2 + Y_*^2 = 1$), and one (unstable) $Y_* = 0$ configuration is possible with the springs compressed ($X^2 + Y_*^2 < 1$).

Since the stability of the $Y_* = 0$ equilibrium changes at the bifurcation point $X_c = 1$, reducing X through $X = 1$ results in the system switching to either one of the stable $Y_* = \pm\sqrt{1 - X^2}$ equilibria, with both springs relaxed at their natural lengths l_0. Crucially, because there is no discontinuity between the stable curves at the critical point $X_c = 1$ (see Fig. 6), the perfect pitchfork bifurcation does not result in a discontinuous critical transition.

5.2. Imperfect pitchfork bifurcation ($0 < |a| < 1$)

To study the imperfect pitchfork bifurcations we suppose that the system is in a stable equilibrium Y_* for some fixed $0 < |a| < 1$, and then consider how the equilibrium configuration changes as X is varied. [It is not necessary to consider situations with $|a| \geq 1$, as these only yield one stable equilibrium (see Fig. 4).] Note that for the purposes of this discussion we shall assume $a > 0$; this can be done without loss of generality because results for $a < 0$ can be deduced from those with $a > 0$ by making the transformation $Y \to -Y$ (due to symmetry).

We obtain an algebraic expression relating Y_* and X from Eq. (3) by solving $A(Y_*; X, a) = 0$ for X, and treating Y_* as the independent variable, that is,

$$X(Y_*; a) = \frac{Y_*}{(Y_* - a)}[1 - (Y_* - a)^2]^{1/2}. \tag{14}$$

The dependence of Y_* on X is then visualised by sketching $X(Y_*; a)$ as in Fig. 7, with Y_* on the ordinate axis; because $X > 0$ and $a \in (0, 1)$, the function $X(Y_*; a)$ only yields physical solutions when either

$$(a-1) < Y_* < 0 \quad \text{or} \quad a < Y_* < (a+1). \tag{15}$$

Notice that Eq. (14) implies

$$X^2 + Y_*^2 = \frac{Y_*^2}{(Y_* - a)^2} \tag{16}$$

so that the springs are compressed ($X^2 + Y_*^2 < 1$) when $Y_* < 0$, and extended ($X^2 + Y_*^2 > 1$) when $Y_* > a > 0$.

Now suppose that the system is in a stable equilibrium configuration with $Y_* < 0$, and that X is increased through $X_c = [1 - a^{2/3}]^{3/2}$. As depicted in Fig. 7, provided X is increased sufficiently slowly (so that perturbations are small), the system remains on the negative stable curve with Y_* increasing continuously until the pitchfork bifurcation point at $X = X_c$ for which $Y_* = -Y_c$. At this point the negative stable curve terminates, and further increases to X force the system to undergo a discontinuous transition to a new stable equilibrium with $Y_* \geq Y_s$ positive. In terms of X_c, the magnitude $\Delta Y_* = Y_s + Y_c$ of this transitions is

$$\Delta Y_* = [1 - X_c^{2/3}]^{1/2}[1 + X_c^{2/3} + (1 - X_c^{2/3} + X_c^{4/3})^{1/2}]. \tag{17}$$

It may be shown that $0 < \Delta Y_* < 2$, with transitions of greatest magnitude ($\Delta Y_* \to 2$) corresponding to $X_c \to 0$, i.e., configurations with the forcing maximised ($a \to 1$).

Notice that although the system shifts discontinuously from $-Y_c$ to Y_s as X is increased through X_c, the converse is not true: once shifted, the system will remain on the $Y_* > 0$ curve irrespective of any changes to X. In this sense catastrophic transitions due to the imperfect-pitchfork bifurcation cannot be reversed.

6. Stability and critical slowing down

In the previous sections we have postulated (albeit informally) strong system damping. Like Ong [13], therefore, we have ignored dynamics, and assumed that disturbances result in the system settling at a point of stable equilibrium. If the damping is sufficiently small, however, then before reaching equilibrium, the system will support transient oscillations similar to those of a free system.

To study this transient behaviour, we neglect damping, and consider the time dependence of small perturbations

$$Y_1(t) = Y(t) - Y_* \tag{18}$$

from a point of equilibrium Y_*, i.e., where $A(Y_*) = 0$. By the first-order Taylor expansion of $A(Y) = A(Y_* + Y_1)$, the normalised total force on this perturbed system is

$$A(Y) \approx A(Y_*) + \left[\frac{\mathrm{d}A}{\mathrm{d}Y}\right]_{Y_*} Y_1 = \left[\frac{\mathrm{d}A}{\mathrm{d}Y}\right]_{Y_*} Y_1, \tag{19}$$

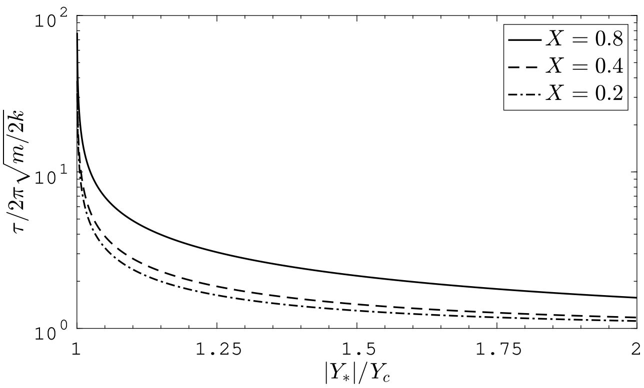

Fig. 8. Oscillation time-period $\tau(Y_*;X)$ as a function of $|Y_*|$ for three illustrative values of $X \in (0,1)$. The time period blows-up ($\tau \to \infty$) as the system approaches bifurcation ($|Y_*|/Y_c \to 1$).

where Eq. (4) yields

$$\left[\frac{\mathrm{d}A}{\mathrm{d}Y}\right]_{Y_*} \equiv A'(Y_*) = -\left[1 - \left(\frac{Y_c^2 + X^2}{Y_*^2 + X^2}\right)^{3/2}\right], \tag{20}$$

with $Y_c = [X^{4/3} - X^2]^{1/2}$. Now, since the system has fixed mass m, Newton's second law $F = m\ddot{y}$ gives

$$A(Y) \equiv \frac{F}{2kl_0} = \frac{m}{2kl_0}\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{m}{2k}\frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} = \frac{m}{2k}\frac{\mathrm{d}^2 Y_1}{\mathrm{d}t^2}. \tag{21}$$

Thus, substituting Eqs. (20) and (21) into (19), we find that perturbations Y_1 satisfy the oscillator equation

$$\frac{\mathrm{d}^2 Y_1}{\mathrm{d}t^2} + \omega^2 Y_1 \approx 0, \tag{22}$$

where ω is given by

$$\omega^2 = \frac{2k}{m}\left[1 - \left(\frac{Y_c^2 + X^2}{Y_*^2 + X^2}\right)^{3/2}\right]. \tag{23}$$

If $\omega^2 > 0$, then Eq. (22) predicts stable oscillations with frequency ω; however, if $\omega^2 < 0$, then perturbations are unstable, and grow exponentially with rate $\gamma = |\omega|$ [14]. Eq. (23) therefore yields the following stability criteria for a given equilibrium configuration Y_*:

$$Y_* \text{ is stable if } |Y_*| > Y_c \text{ and unstable if } |Y_*| < Y_c, \tag{24}$$

where $Y_c = [X^{4/3} - X^2]^{1/2}$. These criteria are consistent with our previous discussions (see, Figs. 5, 6 and 7).

Eq. (23) indicates that the time period $\tau = 2\pi/\omega$ of oscillations about a stable equilibrium is (see Fig. 8)

$$\tau(Y_*;X) = 2\pi\sqrt{\frac{m}{2k}}\left[1 - \left(\frac{Y_c^2 + X^2}{Y_*^2 + X^2}\right)^{3/2}\right]^{-1/2}. \tag{25}$$

Here τ can be used as a way of anticipating when the system is close to a critical discontinuous transition. To see this, recall that when the system reaches a bifurcation point, i.e., as either $|a| \to a_c$ or $X \to X_c$, the equilibrium configuration Y_* tends towards one of the critical values $\pm Y_c$ (see Sections 4 and 5). Crucially, therefore, Eq. (25) predicts that the oscillation period τ diverges as the system approaches bifurcation (see Fig. 8), i.e.,

$$\lim_{|Y_*| \to Y_c} \tau = \infty. \tag{26}$$

This phenomena—whereby the oscillation period blows-up as a system approaches bifurcation—is known as critical slowing down, and has been proposed as an 'early-warning signal' for onset of critical transitions [15].

7. Conclusion

Recently, C. Ong has described a simple V-shaped mass–spring system as an accessible context for illustrating saddle–node bifurcations, and hysteresis [13]. Here we have substantially developed Ong's analysis to show that the same system can be used to demonstrate a much wider range of threshold phenomena, including perfect and imperfect pitchfork bifurcations, and critical slowing down. Indeed, whilst Ong's treatment assumed four control parameters, we have used a dimensionless approach to show that bifurcation is determined by two parameters only: a spring separation parameter X, and a mass forcing parameter a. In this way we have demonstrated that both forms of bifurcation are features of a cusp catastrophe characterised in $X - a$ parameter space by the critical curve $X^{2/3} + a^{2/3} = 1$. What is more, by deriving a linear stability analysis of the system, we have shown how discontinuous transitions can be anticipated by blow-up of the oscillation period near bifurcation (critical slowing down). Given the rich diversity of threshold phenomena exhibited by the system, we look forward to developing a demonstration mass–spring device suitable for future empirical studies.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix. Critical values

Ong's discussion includes a calculation equivalent to determining a_c, but does not include a similarly equivalent computation for Y_s [13]; here we calculate both.

The critical value a_c is the value that a takes when the minimum $-Y_c$ is also a root of $A(Y)$. It therefore follows by Eq. (3) that a_c satisfies

$$A(-Y_c; X, a_c) = a_c + Y_c\left[1 - (X^2 + Y_c^2)^{-1/2}\right] = 0. \tag{A.1}$$

Since $Y_c = [X^{4/3} - X^2]^{1/2}$, we thus have

$$a_c = [1 - X^{2/3}]^{3/2}. \tag{A.2}$$

Hence, the critical equilibrium Y_c may also be written

$$Y_c = [X^{4/3} - X^2]^{1/2} = a_c^{1/3}[1 - a_c^{2/3}], \tag{A.3}$$

where we used Eq. (A.2) to set $X = [1 - a_c^{2/3}]^{3/2}$.

To determine the critical stable equilibrium Y_s we first note that it is a root of $A(Y; a_c)$, i.e.,

$$A(Y_s) = a_c - Y_s\left[1 - (X^2 + Y_s^2)^{-1/2}\right] = 0. \tag{A.4}$$

Since $[1 - (X^2 + Y_s^2)^{-1/2}] < 1$, the fact that Y_s is positive ($Y_s > 0$) means that this equation implies

$$Y_s > a_c. \tag{A.5}$$

Rearranging Eq. (A.4), and expressing X in terms of a_c, we find that Y_s must be a root of the quartic equation

$$Y_s^4 - 2a_c Y_s^3 - 3a_c^{2/3}(1 - a_c^{2/3})Y_s^2 - 2a_c(1 - a_c^{2/3})^3 Y_s + a_c^2(1 - a_c)^3 = 0. \tag{A.6}$$

Thus, factorising the left-hand-side, we require

$$(Y_s + Y_c)^2(Y_s - Y_+)(Y_s - Y_-) = 0, \tag{A.7}$$

where Y_+ and Y_- are given by

$$Y_\pm = a_c^{1/3}[1 \pm (1 - a_c^{2/3} + a_c^{4/3})^{1/2}]. \tag{A.8}$$

It may be shown that $Y_+ > a_c > Y_-$. Hence, given that $Y_s > a_c > 0$, it follows that the only acceptable solution to Eq. (A.7) is

$$Y_s = Y_+ = a_c^{1/3}[1 + (1 - a_c^{2/3} + a_c^{4/3})^{1/2}]. \tag{A.9}$$

Note that when a is fixed, and X is varied, the definition of $a_c = [1 - X^{2/3}]^{3/2}$ means that a is identical to a_c when X takes the critical value

$$X_c = [1 - a^{2/3}]^{3/2}. \tag{A.10}$$

Thus, we may also express the critical equilibria Y_c and Y_s in terms of X_c by setting $a = a_c = [1 - X_c^{2/3}]^{3/2}$, i.e.,

$$Y_c = X_c^{2/3}[1 - X_c^{2/3}]^{1/2}, \tag{A.11}$$

$$Y_s = [1 - X_c^{2/3}]^{1/2}[1 + (1 - X_c^{2/3} + X_c^{4/3})^{1/2}]. \tag{A.12}$$

These are the expressions used to obtain Eq. (17).

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